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Duncan M. Y. Sommerville

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DUNCAN M. Y. SOMMERVILLE.

By GEORGE BRUCE HALSTED.

It has been said the heroic age of non-euclidean geometry is passed, since long gone are the days when Lobachevski flinched into calling his system "imaginary geometry," and, I might add, Gauss kept a more cowardly silence because, as he confesses, he "fears the outcry of the Boeotians," (proverbial for stupidity).

But there are other kinds of heroism beside that typified by the biting blade of Bolyai's sabre, and here I set forth still another paladin, Duncan Sommerville, whose triumph in his *Bibliography of Non-Euclidean Geometry* I can, perhaps, better than any one else appreciate, since I myself wrestled for years at this very labor of Hercules.

Lobachevski was of Kazan, city of the Golden Horde of Tatars on Mother Volga. John Bolyai was born at Kolozsvár, birthplace of Matthias Corvinus, chosen king of Hungary by forty thousand Magyars assembled upon the ice of the frozen Danube, and the young János grew up on the banks of Maros the mordicant. Duncan Sommerville was born in India, in Rajputana, land named from the warrior caste, analogue of the Samurai and like them unorthodox.

In my Biography of Hoüel, of a very old protestant family of Normandy, I say: "The key to his whole mental life was this old protestant blood, which means so much in a Roman Catholic country." The women of his family, who had probably never heard of Sherlock Holmes, wrote me their amazement that I could so have penetrated their family secrets, and used the significant phrase: In science there are no frontiers!

Duncan Sommerville, born November 24th, 1879, was eventually sent to Perth Academy, Scotland, and in 1896 entered St. Andrews University, third Bursar, where he won medals in mathematics, Ordinary and Honors, Greek, botany, and held the Ramsay Scholarship and the Berry Scholarship in mathematics and natural philosophy. In 1900 he graduated with the degree B. Sc. and M. A. with First Class Honors in mathematics and natural philosophy.

He then pursued the first two years of a medical course, but in 1902 began teaching applied mathematics (dynamics). The following year he was appointed assistant to the professor of mathematics at St. Andrews University and lecturer on applied mathematics.

In 1905 he won the degree of D. Sc. for the thesis: "Networks of the Plane in Absolute Geometry," thus using in his thesis title a phrase of my making; for the designation "Absolute Geometry" was first used by myself as a rendering for John Bolyai's phrase *scientiam spatii absolute veram*, in which I have always gloried as showing the magnificent nerve of the young Magyar hero, victim of the meanness of Gauss, as was also his own son who passed his life an exile here in Colorado.

Dr. Sommerville is at present lecturer in mathematics and in applied mathematics at the University of St. Andrews. He is a Fellow of the Royal Society of Edinburgh, and president of the Edinburgh Mathematical Society. He is a member of the Circolo Matematico di Palermo, and of the Perthshire Society of Natural Science.

At the Sheffield meeting of the British Association (1910) at which he read a paper on Non-Euclidean Bibliography, a committee was appointed to consider the advisibility of drawing up a report on the subject of non-euclidean geometry. Dr. Sommerville was made secretary, and Dr. H. F. Baker, of Cambridge, chairman. Other members were: Professor Chrystal of Edinburgh, Dr. A. N. Whitehead of Cambridge (in whose article, Geometry, in the *Encyclopaedia Britannica*, my name occurs thrice), Professor Burnside of Greenwich, and Sir Robert S. Ball of Cambridge. That this Report would be greatly appreciated may be argued from the fact that my own report on non-euclidean geometry to the American Association in 1899, and my supplementary report in 1901, were each published in full four times over, thus vieing in fame with my Bibliography of Hyperspace and Non-Euclidean Geometry, of which the University of St. Andrews advertising in *Science* (No. 865, p. vii) its publication of Sommerville's *Bibliography of Non-Euclidean Geometry* says: "This work is regarded as a continuation of Professor Halsted's often quoted bibliography which was published in the *American Journal of Mathematics.*"

In the session of 1908-9 a mathematical and physical society was instituted in the University of St. Andrews, of which Dr. Sommerville is still president. It exists for the furtherance of an interest in mathematical and physical subjects, and has been very successful, interesting papers being read at fortnightly intervals by members of the staff and by students.

Dr. Sommerville writes me: "I find myself going into the elements of non-euclidean geometry in Honors lectures on geometry, for I think that one cannot arrive at the true nature of ordinary geometry without a comparison with the non-euclidean systems."

How very like is this to the quotation from Professor Eduard Study of Bonn, which I placed at the head of my paper "The Value of Non-

Euclidean Geometry" in *The Popular Science Monthly* for November, 1905: "Among conditions to a more profound understanding of even very elementary parts of the euclidean geometry, the knowledge of the non-euclidean geometry can not be dispensed with."

And now finally of the tremendous achievement to which all this is but a prelude, Sommerville's Bibliography of Non-Euclidean Geometry, suffice it to say that it is an immortal monument, showing to all what we of the cult have ever realized, that here we face one of the very few germinal ideas whose creation has made our modern world.

John Bolyai published but two dozen pages; what I have long called the most extraordinary two dozen pages in the whole history of thought. Here we see shown us a list of their fruit more than four thousand articles and books dealing with this subject.

The following is a bibliography of Sommerville's productions:

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"On the number of independent conditions involved in the vanishing of a rectangular array." Edinburgh, Proc. Math. Soc., 24, 2-6.

1906. "On the distribution of the proper fractions." Edinburgh, Proc. R. Soc., 26, 116-129.

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"Questions 15894, 15903, 15906, 15948, 15925." Math. Questions, *Educational Times*, London, 10, 34, 40, 71, 83-84, 93-94.

1907. "On links and knots in euclidean space of n dimensions." *Messenger Math.*, Cambridge, 36, 139-144.

"On certain projective configurations in space of n dimensions and a related problem in arrangements." Edinburgh, Proc. Math. Soc., 25, 80-90.

"Questions 15986, 15999, 16009." Math. Questions, *Educational Times*, London, 11, 24-25, 55-56, 57-61. [Q. 16009, proposed by H. Bateman, "On systems of hyperspheres, each touching two others in succession." Generalized for euclidean or non-euclidean space of any dimensions.]

1908. "Sunset and twilight curves, and related phenomena" Edinburgh, Proc. R. Soc., 28, 311-337 (3 plates).

"Sunset and twilight." Perth, Proc. Soc. Nat. Sci., 4, clxxxiv-clxxxvi (1 plate).

"Question 15951." Math. Questions, *Educational Times*, London, 12, 24-25.

1909. "On certain periodic properties of cyclic compositions of nnmbers." London, Proc. Math. Soc., (2) 7, 263-313.

1910. "A problem in voting." Edinburgh, Proc. Math. Soc., 28.

"Classification of geometries with projective metric." Edinburgh, Proc. Math. Soc., 28, 25-41.

"Elementary considerations relating to the absolute." Edinburgh, Proc. Math. Soc., 28, 65-72.

"Note on the geometries in which straight lines are represented by circles." Edinburgh, Proc. Math. Soc., 28, 81-94.

"The early history of non-euclidean geometry." *Nature*, London, 84, 172.

"On the need of a non-euclidean bibliography." Rep. Brit. Assoc., 80 (Sheffield), 531-532.

1911. "The numerically greatest term of a binomial expansion." Edinburgh, Math. Notes, No. 7, 74-77.

"Concrete representations of non-euclidean geometry." In a Collection of Scientific Papers published by the University of St. Andrews to commemorate the 500th anniversary of its founda-

"Bibliography of non-euclidean geometry, including the theory of parallels, the foundations of geometry, and space of n dimensions." London: Harrison, pp. 400.

NOTES ON GREATEST COMMON DIVISOR AND LEAST COMMON MULTIPLE OF INTEGERS.

By BENJAMIN FRANKLIN YANNEY, Wooster University.

I. Properties of Quotients.

Let $q_1, q_2, \ldots, q_n$ be the respective quotients obtained by dividing the positive integers $a_1, a_2, \ldots, a_n$ by their greatest common divisor D; and $q'_1, q'_2, \ldots, q'^n$ the respective quotients obtained by dividing L, the lowest common multiple of the integers $a_1, a_2, \ldots, a_n$, by the integers, in turn. Hence

$$q_1 = a_1 \div D;\ q_2 = a_2 \div D;\ \ldots;\ q_n = a_n \div D \ldots (1);$$
$$q'_1 = L \div a_1;\ q'_2 = L \div a_2;\ \ldots;\ q'_n = L \div a_n \ldots (2).$$

The two sets of quotients possess the following properties:

1. Neither set of quotients has a factor common to all the numbers in the set. For, otherwise, D would not be the greatest common divisor nor L the least common multiple of the numbers $a_1, a_2, \ldots, a_n$.

2. $q_1 q'_1 = q_2 q'_2 = \ldots = q_n q'_n = L \div D$.

This result is obtained by multiplying, member by member, the cor-

responding equalities in (1) and (2). Furthermore, by suitably combining all the equalities in (1) and (2), we get

3. $(q_1q_2\ldots q_n)(q'_1q'_2\ldots q'_n)=L^n\div D^n$; and,

4. $(a_1a_2\ldots a_n)^2=L^nD^n(q_1q_2\ldots q_n)\div(q'_1q'_2\ldots q'_n)$.

From the second property stated above it is easily seen that

5. $L\div D$ divided in order by either set of quotients gives in corresponding order the other set.

From the first and fifth properties, it follows that

6. The least common multiple of each set of quotients is $L\div D$.

II. Relation of Integers to their Greatest Common Divisor and Least Common Multiple.

1. As is well known, the product of two positive integers, a_1 and a_2, is equal to the product of their greatest common divisor D and least common multiple L:

$$a_1a_2=DL.$$

The general relation may be stated as follows:

$$DL^{n-1}\geqq a_1a_2\ldots a_n\geqq D^{n-1}L\ldots\text{(A)}.$$

Proof. From (2) of I, we get

$$L^n=(a_1a_2\ldots a_n)(q'_1q'_2\ldots q'_n).$$

But, as previously shown, the least common multiple of q'_1, q'_2, ..., q'_n is $L\div D$. Hence, $q'_1q'_2\ldots q'_n\geqq L\div D$. Therefore, $L^n\geqq(a_1a_2\ldots a_n)L\div D$.

$\therefore DL^{n-1}\geqq a_1a_2\ldots a_n$.

Similarly, starting with (1) of I, it can be shown that

$$a_1a_2\ldots a_n\geqq D^{n-1}L.$$

2. Another form of the general case is as follows:

$$\prod {}_kD_i\prod {}_kL_i^{k-1}\geqq(a_1a_2\ldots a_n)^{\frac{(n-1)!}{(n-k)!(k-1)!}}\geqq\prod {}_kD_i^{k-1}\prod {}_kL_i,\ldots\text{(B)},$$

in which ${}_kD_i$ and ${}_kL_i$ represent, respectively, the greatest common divisors and least common multiples of the n integers taken k at a time, i thus being

equal to 1, 2, ..., $\frac{n!}{(n-k)!k!}$.

The proof of this relation follows easily from (A), which is true for each set of k integers selected from the entire set of n integers. There will evidently be, in all, $\frac{n!}{(n-k)!k!}$ such relations. Multiplying together these relations, member by member, and noticing that in the middle member each of the n integers will occur $\frac{(n-1)!}{(n-k)!(k-1)!}$ times, the desired result is obtained.

3. If in (B) we set $k=2$, we obtain

$$(a_1 a_2 \ldots a_n)^{n-1} = \prod {}_2D_i \prod {}_2L_i, \quad i=1, 2, \ldots, \tfrac{1}{2}n(n-1),$$

which is probably the most general equality relation existing between integers, and greatest common divisors and least common multiples.

If, finally, $n=2$, we arrive at the well-known relation

$$a_1 a_2 = DL.$$

A POINT'S VISIT TO THE LINEAR CONTINUUM.

By H. W. REDDICK, Columbia University.

Once upon a time a point, XYZ, in three-dimensional space, decided to visit the set of points on a certain straight line. He had a sense of his superiority over his less fortunate fellow-beings who were compelled to lead a one-dimensional existence, but his motiqe for visiting them might be called philanthropic for he had a real desire to find out the relations existing between them and to enlighten them, if possible, concerning a higher existence.

As XYZ approached the line he was surprised to find that its residents were not living happily together; and he soon learned that they were capable of the same petty dissensions and jealousies which he had supposed were possible only in the enlightened set to which he belonged. He found that none of them was on speaking terms with his next-door neighbor — in fact, that he did not know who his next-door neighbor was. The society was divided up into political parties, religious denominations, and exclusiqe cliques, which spent most of their time quarrelling with one another, and did not seem to realize that, taken all together, they formed one Grand Continuum.

The society was divided into two great political sets, the Rationalists and the Irrationalists. The members of the Rational Set believed that the tariff on any commodity should be expressible in a finite number of figures. Their opponents could see no reason for such a belief. The chief argument of the Rationalists against the Irrationalists was that they were irrational. It was impossible, however, to settle any question by voting because there was always trouble in counting the Irrational vote.

There were two great religious denominations, the Transcendental and the Algebraic sects. The Transcendentalists looked upon those of the opposite faith with a feeling of pity, not unmingled with contempt, reproaching them for allowing themselves to be counted and for serving as roots of mere algebraic equations with rational coefficients. But the members of the Algebraic Set were firm in their pragmatic belief and denounced their opponents as being unpractical. Every inhabitant of the Continuum felt that he had the right of membership in one of these religious denominations but there was a great multitude who did not know to which set they should belong. However, there were two saints among the Transcendental Set whose right to be termed orthodox members of the faith had been established beyond doubt. These were e and π. Of these, e was the patron saint, who had been canonized by a human, Hermite, in 1873. e had become distinguished, however, long before this, by being appointed to act as base for the Napierian system of logarithms. But the members of the Transcendental Set never mentioned this distinction when eugolizing e in the presence of a member of the Algebraic Set, for the latter would promptly remind them that one of his number, 10, had achieved a greater glory by holding the office of base of the more practical Briggs system of logarithms. The other saint, π, had come up through great tribulation. Many were the mathematical sins that had been committed in his name! Owing to his pecaliar attribute of being the ratio of a circle to its diameter, he had often been misunderstood and cursed as a member of the Algebraic Set. But his vindication, glorious and complete, was brought about in 1882 by Lindemann. *XYZ* found that the two following questions were the ones most discussed by the Transcendentalists: Which of our number will be the next to be proved worthy of membership? Will a rule ever be found that can be applied to any individual to determine to which set he should belong?

Mr. *XYZ* then turned his attention to some of the more exclusive cliques that had recently been formed. There were several sets, regarded as snobbish by the others, who called themselves perfect, but they were all closed, so that *XYZ* failed to obtain much information about them. One of the most famous of these was Cantor's Typical Ternary Club, a very aristocratic organization whose members looked upon all non-members as the masses who were everywhere dense. The latter, however, referred sneeringly to the members of the organization as of content zero.

Mr. *XYZ* then dropped in at the Integral Club, where a number of in-

tegers had assembled. One of them informed *XYZ* that their members had been added together and had multiplied rapidly and in some instances one had been subtracted from another or a family had been divided, but that in no case had a number been produced which did not belong to the set. "However, we are a discreet set," he added with a smile. "We are careful not to let the root-extractor enter our midst." The members then began discussing their figures, the figures by which they were recognized. "I maintain that I have the most peculiar shape," said 6, "See! when I stand on my head I become 9." "That 's nothing, said 8, "When I lie down I become infinite." "But then you are no longer one of us," replied the others, "What shall we do with him for perpetrating such a joke? We 'll leave it to you, Mr. *XYZ*." "Well," said *XYZ*, "I think he should be killed. Let him that is greatest among you cast the first stone." They were all silent. Then Zero rolled along and tried to start an argument. "You do n't amount to anything anyhow," the others protested. "Just the same," retorted Zero, "I claim a distinction of which none of you can boast: a distinguished human, Mr. Russell, has dedicated a chapter to me."

At this stage Mr. *XYZ*, regarding his mission as hopeless, moved away into space, more firmly grounded in his conviction that the Continuum will never be well-ordered.

AN EXAMPLE OF THE USEFULNESS OF FOURIER'S THEOREM IN SEPARATING THE ROOTS OF EQUATIONS.

By L. R. MANLOVE, England.

Cases must occur in practice where the roots of an equation cannot be separated by any of the well-known easier methods and where Sturm's Theorem is inapplicable by reason of the labor which it involves. In such cases resort may be had to a combination of Fourier's Theorem with Lagrange's method of approximation, as shown below.

EXAMPLE.

Let us take the equation

$$x^{17}-35x^{15}+11x^{14}-1000x^{10}+2500x^5-151x^3+1=0.$$

A first application of Fourier's Theorem shows that there are:

(a) *Two positive roots:* One between 1 and 2; one between 5 and 6.

(b) *Three negative roots:* One between 0 and -1; one between -1 and -2; one between -6 and -7.

(c) *A doubtful interval* between 0 and +1 in which 4 changes of sign are lost and which *may* consequently include 4 more positive roots.

To dispose of the doubtful interval, put $x = \frac{1}{u_1}$.

The equation in u_1 is:

$$u_1^{17} - 151u_1^{14} + 2500u_1^{12} - 1000u_1^7 + 11u_1^3 - 35u_1^2 + 1 = 0 = \text{(say)}\, F_A(u_1).$$

Fourier's functions of $F_A(u_1)$ give for the value 1 of u_1 two changes of sign, and for the value 2 no changes of sign.

$F_A(u_1) = 0$ may therefore have two roots between 1 and 2, but has no other positive roots greater than unity.

The four originally doubtful roots are reduced to two.

To dispose of the remaining pair put $u_1 = 1 + \frac{1}{u_2}$.

The equation in u_2 is:

$$1327u_2^{17} + 20866u_2^{16} + \text{(9 positive terms)} - 12588u_2^6 - 5053u_2^5$$
$$+ \text{(5 positive terms)} = 0 = \text{(say)}\ F_B(u_2),$$

which clearly can have no positive root greater than unity.

The four originally doubtful roots are all imaginary and the proposed equation is disposed of.

REMARKS.

The formation (by Horner's method) of the equation $F_B(u_2) = 0$ is effected *currente calamo* and the whole example might probably be worked in less than an hour.

It will be observed that in the auxiliary equations used in this method we are only concerned with positive values of the variable greater than unity.

Negative roots of the original equation are most conveniently dealt with by substituting $-y$ for x in that equation and seeking the positive roots of the resulting equation in y.

The writer has tested the method by a number of trials and finds that for most intervals the doubtful roots are at or before the second auxiliary equation, either separated or shown to be imaginary, and in no case, so far, has he found it necessary to go beyond the third auxiliary equation.

Failure to attain definite results within the limits just specified creates a presumption of the existence of equal roots which must be dealt with in the ordinary way before proceeding farther.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

358. **Proposed by V. M. SPUNAR, M. and E. E., Chicago, Illinois.**

Show that $\frac{n(n+1)\dots(n+m-1)}{m!}-n\frac{n(n+1)\dots(n+m-4)}{(m-3)!}+\frac{n(n-1)}{2!}\cdot\frac{n(n+1)\dots(n+m-7)}{(m-6)!}-\dots=0$, when $m>2n$; and $=1$ when $m=2n$.

Solution by B. F. FINKEL, Ph. D., Drury College.

We have $(1-x)^{-n}=1+nx+\frac{n(n+1)}{1.2}x^2+\frac{n(n+1)(n+2)}{1.2.3}x^3+\dots$

$$+\frac{n(n+1)\dots(n+m-1)}{m!}x^m+\dots$$

Also, $(1-x^3)^n=1-nx^3+\frac{n(n-1)}{1.2}x^6-\frac{n(n-1)(n-2)}{1.2.3}x^9+$

$$\dots+(-1)^m\frac{n(n-1)(n-2)\dots(n+m-1)}{m!}x^{3m}+\dots$$

The series under consideration is the coefficient of x^m in the product of the right-hand members of these two equations, which is the same as the coefficient of x^m in the product of $(1-x)^{-n}(1-x^3)^n$ or $(1+x+x^2)^n$.

Now the coefficient of x^m in the expansion of $(1+x+x^2)^n$, by the *multinomial theorem*, is

$$\Sigma\frac{n!}{\alpha!\,\beta!\,\gamma!},\text{ where }\alpha+\beta+\gamma=n\text{ and }\beta+2\gamma=m.$$

The solutions of these two equations are

$\gamma=0,\ 1,\ 2,\ 3,\ \dots,\ n;$

$\beta=m,\ m-2,\ m-4,\ m-6,\ \dots,\ m-2n;$

$\alpha=n-m,\ n-m+1,\ n-m+2,\ n-m+3,\ \dots,\ 2n-m.$

$$\therefore \frac{n(n+1)(n+2)\dots\dots(n+m-1)}{m!} - n\frac{n(n+1)\dots(n+m-4)}{(m-3)!}$$

$$+\frac{n(n-1)}{1.2.3}\frac{n(n+1)\dots(n+m-7)}{(m-6)!}\dots$$

$$=\frac{n!}{(n-m)!\,m!}+\frac{n!}{(n-m+1)!\ (m-2)!\ 1!}+\frac{n!}{(n-m+2)!\ (m-4)!\ 2!}+\dots$$

$$+\frac{n!}{(2n-m)!\ (m-2n)!\ n!}.$$

Now, if $m>2n$, there are no terms in the expansion of $(1+x+x^2)^n$ of which the exponent of x is m. Hence, the second member of the above equation is zero. When $m=2n$, there exists only one term of which the exponent of x is $m=2n$, and its coefficient is the last term of the above equation, which is equal to 1.

The fact that there are no terms of which the exponent of x is m when $m>2n$ is manifested also by some of the signs in the factors of the denominators becoming negative.

359. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Illinois.

Show when $1/(1-x)(1-x^3)(1-x^5)(1-x^7)\dots=(1+x)(1+x^2)(1+x^3)(1+x^4)\dots$

Solution by B. F. FINKEL, Ph. D., Drury College.

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\dots(1-x^{2n-1})}$$

$$\equiv\frac{(1+x)(1+x^2)(1+x^3)\dots(1+x^n)}{(1-x)(1+x)(1+x^2)(1-x^3)\ (1+x^3)\ (1+x^4)\dots(1-x^{2n-1})}$$

$$\equiv\frac{(1+x)(1+x^2)(1+x^3)\dots(1+x^n)}{(1-x^{2^{p}-1})(1-x^{3.2^{q}-1})(1-x^{5.2^{r}-1})(1-x^{7.2^{s}-1})\dots\dots(1-x^{2n-1})},$$

where $p+q+r+s+\dots=4n-4$,

$$\doteq(1+x)(1+x^2)(1+x^3)\dots(1+x^n)\dots \text{ as } n\doteq\infty,$$

when $-1<x<1$, since, when $-1<x<1$, each factor in the denominator approaches 1.

NOTE. The editors desire that contributors send in good problems for solution. Let us have a great variety of problems for solutions in the various departments. Also send in solutions; we prefer to publish solutions prepared by contributors, rather than publish our own. We have neither time, inclination, nor ability to solve every problem proposed in the MONTHLY. However, if every contributor will give a little time to the problems each month, by united effort, there will be few problems remain unsolved. We have recently republished a number of unsolved problems, and we shall be pleased to have solutions of any of them, or any others that remain unsolved. ED, F.

GEOMETRY.

384. **Proposed by S. LEFSHETZ, Universityof Nebraska.**

Let ABC be a triangle, O a circle tangent to its three sides, T a variable tangent of O, which cuts the sides BC, CA, AB in a, b, c. Oa', Ob', Oc' the perpendiculars in O to Oa, Ob, Oc, cutting, respectively, T in points a', b', c'. Prove that Aa', Bb', Cc' meet in a point t, and find the locus of t when T varies. Purely geometrical proofs wanted.

Solution by R. P. BAKER, Iowa City, Iowa.

I. By elementary geometry.

Lemma I. The second tangent from a', $a'K$ to the circle is parallel to BC.

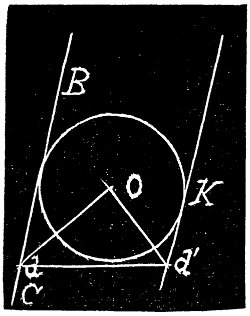

Fig. 1.

For $a'aB$, $aa'A$ are the doubles of the complementary angles $a'aO$ and $aa'O$, and hence supplementary. So $a'K$ is parallel to aB, that is to CB.

Lemma II. If the tangents to a circle from P meet two parallel tangents in Q, Q'; R, R', respectively, and O is the center; then $PQ.PR=OP^2=PQ'.PR'$.

Let M, N be the points of contact of PQ, PR, and S, T of the parallel tangents.

Then $OPQ=OPR$; $2OQP=$supp. $SOM=$supp. $(SOP-MOP)=$supp. $SOP+MOP$.

$2POR=2POT-TON=2POT-(POT-NOP)=POT+MOP=$supp. $SOP+MOP$. Therefore, $OQP=POR$, and the triangles OPQ and RPO are similar.

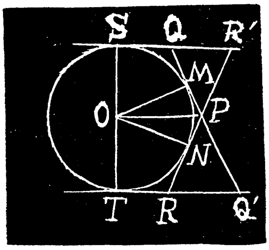

Fig. 2.

Hence $PQ.PR=OP^2$, and by the similar triangles PRQ', $PR'Q$, each is equal to $PQ'.PR'$.

Applying the lemmas to the figure, we have $ab'.aC=ac'.aB$, and the triangles aBb', aCc' are similar, having equal angles at a, and Bb', Cc' are parallel. So for Aa', Bb'.

II. By Brianchon's Theorem.

If the second tangents from b', c' meet at A', we have in AB, BC, CA, $b'A'$, bc, $c'A'$ six tangents to a conic. If the lines are taken in the order written the joins of the cuts of opposite pairs are Bb', Cc', and the line at infinity. Hence Bb', Cc' are parallel, and similarly, the other pairs.

III. Consider the locus of intersections of Bb' and Cc. The pencils at B and C are projective, being in $1:1$ correspondence with the tangents T by a ruler construction. The intersection is at infinity on AB when T coincides with AB, and at infinity on AC when T coincides with AC. It is on BC only if T coincides with BC. From the latter fact the locus is a straight line, and from the two former the line at infinity, the pencils being in perspective.

IV. The generalized problem.

AB, BC, CA, Ta, OI, OJ are six tangents to a conic: Ta cuts AB in c, BC in a, CA in b; OI, OJ have chord of contact IJ: Oa' is the harmonic conjugate of Oa with respect to OI and OJ: a' is its cut with Ta, and so for Ob', Oc'. Then $Aa'Bb'Cc'$ are concurrent on IJ.

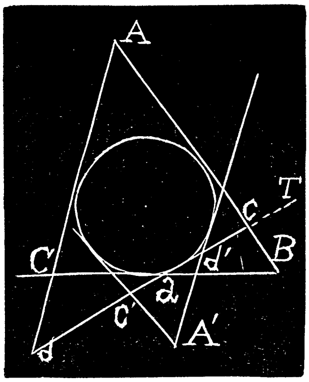

Fig. 3.

Let AB have contact at γ, and OI at κ, IJ at P, OJ at λ. Let PG be the second tangent from P, G its point of contact. $G\gamma$, the polar of P, passes through O, the pole of PIJ. Let it cut IJ at π. Then $PI\pi J$ is harmonic, and so is $P\kappa\gamma\lambda$. The latter is the range determined on the tangent AB by the four tangents PG, OI, AB, OJ. The range on ab being equal, PG cuts this line in c'.

This proves Lemma I in general. Brianchon's Theorem is then applied as in II.

V. By Analytic Geometry.

Take the equation of the circle as

$$x=\frac{1-\lambda^2}{1+\lambda^2};\quad y=\frac{2\lambda}{1+\lambda^2}.$$

The tangent at λ is

$$(1-\lambda^2)x+2\lambda y=1+\lambda^2$$

and intersects the tangent at κ in

$$x=\frac{1-\kappa\lambda}{1+\kappa\lambda};\quad y=\frac{\kappa+\lambda}{1+\kappa\lambda}.$$

The line through the center perpendicular to the join of this point and the center is:

$$x(1-\kappa\lambda)+y(\kappa+\lambda)=0,$$

and cuts the κ tangent in

$$x=\frac{\kappa+\lambda}{\lambda-\kappa};\ y=\frac{\kappa\lambda-1}{\lambda-\kappa}.$$

The line joining this to the cut of tangents at (μ, ν) has the slope

$$\frac{\kappa(\lambda+\mu+\nu+\lambda\mu\nu)-(1+\lambda\mu+\mu\nu+\nu\lambda)}{2(\kappa+\lambda\mu\nu)}$$

which is symmetrical in (λ, μ, ν). The three lines given by interchanges of λ, μ, ν are therefore parallel.

CALCULUS.

312. Proposed by C. N. SCHMALL, New York City.

Given $y^3-3y+x=0$, prove by Maclaurin's theorem, that

$$y=\frac{x}{3}+\frac{x^3}{3^4}+\frac{x^5}{3^6}+\text{etc.}$$

I. Solution by H. PRIME.

Put $u=y^3-3y+x=0$. Then $\partial u/\partial x=1$, $\partial u/\partial y=3y^2-3$; hence, $dy/dx=\frac{1}{3}(1-y^2)=(1+y^2+y^4+y^6+\text{etc.})/3$.

$$d^2y/dx^2=(2y+4y^3+6y^5+\text{etc.})(1+y^2+y^4+\text{etc.})/3^2$$
$$=(2y+6y^3+12y^5+\text{etc.})/3^2.$$

$$d^3y/dx^3=(2+18y^2+60y^4+\text{etc.})(1+y^2+y^4+\text{etc.})/3^3$$
$$=(2+20y^2+80y^4+\text{etc.})/3^3.$$

$$d^4y/dx^4=(40y+320y^3+\text{etc.})(1+y^2+y^4+\text{etc.})/3^4$$
$$=(40y+360y^3+\text{etc.})/3^4.$$

$$d^5y/dx^5=(40+1080y^2+\text{etc.})(1+y^2+y^4+\text{etc.})/3^5$$
$$=(40+1120y+\text{etc.})/3^5;\ \text{etc.}$$

When $x=0$, $(y)=0$. Hence

$(dy/dx)=\frac{1}{3}$, $(d^2y/dx^2)=0$, $(d^3y/dx^3)=2/3^3$, $(d^4y/dx^4)=0$, $(d^5y/dx^5)=40/3^5$, etc. By Maclaurin's formula, $y=x/3+x^3/3^4+x^5/3^6+\text{etc.}$

II. **Solution by A. M. HARDING, University of Arkansas, and JAMES A. BULLARD, Worcester Polytechnic Institute.**

Let $f(x, y) \equiv y^3 - 3y + x = 0$. Then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{1}{3y^2-3} = f'(x, y); \quad \frac{d^2y}{dx^2} = \frac{-6y}{(3y^2-3)^3} = f''(x, y);$$

$$\frac{d^3y}{dx^3} = \frac{-6(15y^2+3)}{(3y^2-3)^5} = f'''(x, y); \quad \frac{d^4y}{dx^4} = \frac{-24.45y(2y^2+1)}{(3y^2-3)^7} = f^{\text{iv}}(x, y);$$

$$\frac{d^5y}{dx^5} = \frac{-120.9(56y^4+57y^2+3)}{(3y^2-3)^9} = f^{\text{v}}(x, y); \ldots$$

Substituting the values of these derivatives for $x=0$, *i. e.* when $y=0$, in Maclaurin's formula, we have

$$y = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{\text{iv}}(0)}{4!}x^4 + \frac{f^{\text{v}}(0)}{5!}x^5 + \ldots$$

$$= \frac{x}{3} + \frac{x^3}{3^4} + \frac{x^6}{3^6} + \ldots$$

NOTE. When $x=0$, $y=0$, $\sqrt{3}$, or $-\sqrt{3}$. Then there are two other series for y obtained by substituting $y=\sqrt{3}$ and $y=-\sqrt{3}$ in the above derivations.

Solved similarly by V. M. Spunar, S. G. Barton, M. E. Graber, and C. N. Schmall.

313. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Evaluate the definite integral $\int_0^\infty (e^{-2ax^2} + e^{2ax^2})\,dx$.

Solution by C. N. SCHMALL, New York City.

$$\int_0^\infty (e^{-2ax^2} + e^{2ax^2})\,dx = \int_0^\infty e^{-2ax^2}dx + \int_0^\infty e^{2ax^2}dx.$$

The first of these two integrals can be evaluated by meams of the familiar result $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$ (see Williamson's *Integral Calculus*).

Putting $x\sqrt{(2a)}$ for x, we have

$$\int_0^\infty e^{-2ax^2}dx = \tfrac{1}{2}\sqrt{\frac{\pi}{2a}}.$$

Now, the expression e^{2ax^2} can be integrated only, so far as we know, by expanding it in an infinite series, thus:

$$e^{2ax^2}=1+2ax^2+\frac{4a^2x^4}{1.2}+....$$

$$\therefore \int e^{2ax^2}dx=x+\frac{2ax^3}{1.3}+\frac{4a^2x^5}{1.2.5}+....$$

NOTE. Professors Harding and Prime should have received credit for solving 308.

PROBLEMS FOR SOLUTION.

ALGEBRA.

366. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Eliminate m from the equations

$$3a^2m^4+4aym^3+6axm^2-(x^2+y^2-4ax)=0;$$
$$(x^2+y^2-4ax)m^4-6axm^2+4aym-3a^2=0.$$

367. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Michigan.

Solve the simultaneous equations:

$$\frac{2x}{1+x^2}=y\ldots(1);\quad \frac{2y}{1+y^2}=z\ldots(2);\quad \frac{2z}{1+z^2}=u\ldots(3);\quad \frac{2u}{1+u^2}=x\ldots(4).$$

368. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Michigan.

Solve the functional equation, $\dfrac{f(-x)}{f(x)}=r^{2x}$.

GEOMETRY.

399. Proposed by J. K. ELLWOOD, Superintendent of Schools, Lucas, Kansas.

A race track is to be composed of two tangents and the arc of the circle which is concave towards the point of intersection of the two tangents, each tangent and the arc of the circle being 1 mile. What is the radius of the circle?

400. Proposed by FRANCIS RUST, C. E., Pittsburgh, Pennsylvania.

Given a circle and a point P without; construct, *using the straight edge only*, the two tangents to the circle through P.

CALCULUS.

320. Proposed by J. F. LAWRENCE, Stillwater, Oklahoma.

Show that, if $u=1+A_1x+\frac{1}{2\,!}A_2x^2+\frac{1}{3\,!}A_3x^3+\ldots$ where the quantities A are connected by the relation $A_m=mA_{m-1}-\frac{1}{2}(m-1)(m-2)A_{m-3}$, then $\log[u(1-x)^{\frac{1}{2}}]=\frac{1}{2}x+\frac{1}{4}x^2$. [From Forsyth's *Differential Equations*, p. 48.]

321. Proposed by ARTEMAS MARTIN, Ph. D., LL. D., United States Coast and Geodetic Survey Office, Washington, D. C.

To a person in a boat at the center of a circular pond the bottom appears to be perfectly level. What is the actual form of the bottom of the pond, the depth of the water at the center being a feet, and the distance of the eyes of the observer from the surface of the water being b feet. [From the *Mathematical Visitor*, Vol. 2, No. 2, p. 62.]

MECHANICS.

267. Proposed by PROFESSOR G. H. LIGHT, Purdue University, Lafayette, Indiana.

A parabolic curve is placed in a vertical plane with its axis vertical and vertex downwards, and inside it, and against a peg in the focus, and against the concave arc, a smooth uniform and heavy beam rests; required the position of equilibrium. [From Bowser's *Mechanics*, Ex. 37, p. 96.]

268. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

ABC and ADE are two uniform congruent wedges, each of weight w. $B=D=90°$. At B and D they are smoothly hinged to a horizontal table. The bases AB, AC all but meet at A, the common foot of two rough inclined faces AC, AE. A rod, weight W, length $2l$, reclines horizontally and symmetrically with an end on each inclined face. Find the conditions of equilibrium.

NOTES AND NEWS.

The winter meeting of the Chicago section of the American Mathematical Society was held in Chicago on Friday and Saturday, December 28, 29, 1911. There were fifty-seven in attendance upon the various sessions, including forty-three members of the Society. Seventeen papers were read in the three sessions and a most enjoyable time was spent at the dinner on Friday evening at the Quadrangle Club. S.

Professor J. McKeen Cattell, Columbia University, lectured, on January 22, before the Senate of the University of Illinois, on the subject of University Administration. Professor Cattell holds the view that administrative officers, including the President of the University, should not receive any larger salaries than the most competent professors are paid. He would also put the emphasis on good men rather than on fine buildings and grounds. M.

In a series of articles under the title "Twelve Major Prophets of Today," now appearing in *The Independent*, Henri Poincaré, the eminent French mathematician is the subject of the third paper. The author, Dr. Edwin E. Slosson, pays the following tribute to one of the *Monthly's* contributors: "In this country Poincaré has become known largely through the efforts of Professor George Bruce Halsted of the State Normal School of Greeley, Colorado, who has translated his philosiphical works and has for many years been indefatigable in spreading the new gospel of the non-euclidean geometry. S.

The sixty-third meeting of the American Association for the Advancement of Science was held at Washington, D. C., December 27 to December 30, and was attended by about 2500 members of the Association and affiliated societies. At this meeting Professor Moore gave his retiring address as chairman of Section A. His subject was, "On the foundation of the theory of linear integral equations. Professor VanVleck, University of Wisconsin, was elected chairman of Section A for the next meeting, which is to be held at Cleveland, Ohio, during the week in which January first, 1913, falls. M.

The United States Bureau of Education has recently issued Bulletins Numbers 13 and 16 for 1911, the former containing the Report of the American Committees I and II, on Mathematics in the Elementary Schools of the United States; and the latter containing the Report of the American Committees III and IV, on Mathematics in the Public and Private Secondary Schools. These Reports are prepared under the direction of the American Commissioners of the International Commission on the Teaching of Mathematics. They may be secured gratis by addressing the United States Commissioner of Education at Washington. S.

At the annual meeting of the Central Association of Science and Mathematics Teachers held at Lewis Institute in Chicago, there were presented at the Mathematics section two important reports of committees: (1) on Results of Mathematical Teaching, Ascertained by Scientific Spirit in the Daily Work and by Scientific Methods of Testing Efficiency; and (2) on Uniform Notation for Algebra. The first report was presented by Professor C. E. Comstock of Bradley Polytechnic Institute, Peoria, Illinois, and and was discussed by Mr. Charles Otterman, of Cincinnati, Ohio, and the second was presented by Mr. L. P. Jocelyn of Ann Arbor, Michigan, High School, and was discussed by Professor E. R. Hedrick, of the University of Missouri, and by Professor George R. Twin, of Ohio State University. An interesting paper also was presented by Mr. K. G. Smith, of the University of Wisconsin on, "The Applications of Mathematics to Problems of the Shop." The following officers were elected for the year 1912; Chairman, Ira S. Condit, State Teachers College, Cedar Falls, Iowa; Vice-Chairman, Charles W. Newhall, Shattuck School, Faribault, Minnesota; Secretary, Miss Marie Gugle, Central High School, Toledo, Ohio. S.

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SOME MECHANICAL DEVICES TO GENERATE CERTAIN SYSTEMS OF CURVES.

By ARNOLD EMCH, University of Illinois.

1. The construction of curves and surfaces by mechanical means, especially by linkages, has been the object of a number of investigations. Some of the most important results along these lines are due to Kempe* and Koenigs.† The first proved that all algebraic curves in a plane may be generated by link-motions. Koenigs found the same to be true for algebraic curves in space. In a further generalization the writer has shown‡ that every algebraic transformation between any number of variables, and as a consequence between any number of complex variables, may be realized by linkages. In another place§ I have, in particular, described linkages for continuous groups of collineation in a plane. These imply the description of conics.

Based upon an articulated regulus‖ it is possible to perform by link-motions certain transformations of a plane into particular quadrics and quartic surfaces and to describe twisted curves of higher order.¶

There are, however, a great number of other mechanical and physical devices besides link-motions by which various systems of curves, for example Lissajous curves in acoustics, may be generated. The following simple examples, in which again ruled quadrics appear, may be of interest in this respect.

2. Consider a variable string model of a regulus in which strings of equal length are used. One end of each string is attached to a point A' of the directrix X', while the other extremity of the string is allowed to slip freely through the corresponding point A of a second directrix X and is kept

* "How to draw a straight line," 1877.

† *Lecons de Cinematique*, 1897, pp. 302-307.

‡ *Transactions of the American Mathematical Society*, Vol. III (1902), pp. 493-498.

§ *Introduction to Projective Geometry and Its Applications*, 1905, pp. 242-260.

‖ A regulus is a ruled quadric.

¶ *Kinematische Gelenksysteme und die durch sie erzeugten geometrischen Transformationen*, Solothurn, 1906. *Archives der sciences physiques et naturelles*, Vol. 24 (1907), p. 368.

taut by a weight at its end P, so that, no matter what the relative position of X' and X may be, AP is always in a vertical position.

If the set of all points A is projective to the corresponding set A', then, as is well known, the strings represent one of the systems of generatrices of a regulus, while X and X' are two elements of the other system. An infinite number of reguli are obtained by changing the relative position of X and X' in space.

Assuming a general projective relation between the points X and X', then the locus of all points P is a quartic.

To prove this proposition, X may be assumed through the origin and in the $\xi\zeta$-plane of a rectangular system of coördinates in space and X' as a line parallel to the ζ-axis and intersecting the η-axis at a distance p from the origin. This can be done without loss of generality. In Fig. 1, let A and A' be two corresponding points on X and X', and let $OA=x$ and $O'A'=x'$, then

$$x'=\frac{ax+b}{cx+d}.$$

If B is the projection of A' on the $\xi\zeta$-plane, and ϕ the angle which the projections of X' upon the same plane includes with X, then from the figure, $(A'A)^2=(AB)^2+(A'B)^2$. But $(AB)^2=(OA)^2+(OB)^2-2OA.OA\cos\phi=x^2+\left(\frac{ax+b}{cx+d}\right)^2-2x\frac{ax+b}{cx+d}\cos\phi$,

hence,

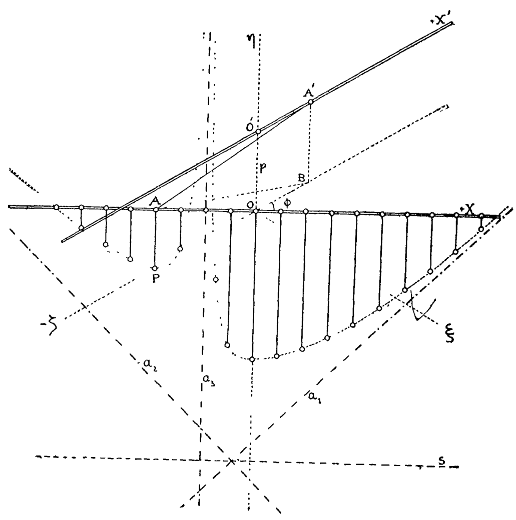

Fig. 1.

$$(A'A)^2=x^2+\left(\frac{ax+b}{cx+d}\right)^2-2x\frac{ax+b}{cx+d}\cos\phi+p^2.$$

Now $A'A+AP$ equals the constant length l of the strings, so that $AA'=l-AP$, or designating the ordinate of the required curve by $AP=y$,

$$(1)\quad (l-y)^2=x^2+\left(\frac{ax+b}{cx+d}\right)^2-2x\frac{ax+b}{cx+d}\cos\phi+p^2,\ \text{or}$$

$$[x^2-(l-y)^2+p^2](cx+d)^2-2x(ax+b)(cx+d)\cos\phi+(ax+b)^2=0.$$

This is the equation of the required curve, which is of the fourth order and

which depends on two variable parameters p and ϕ. By changing these arbitrarily a two parameter system of quartics is obtained. The quartics corresponding to all other positions of X and X' result from linear transformations of the first system.

It can easily be established that all curves of system (1) have the three lines a_1, a_2, a_3 with the equations

$$y=x+10-\frac{a}{c}\cos\phi, \quad y=-x+10+\frac{a}{c}\cos\phi, \quad x=-\frac{d}{c}$$

as common asymptotes. In Fig. 1, which represents an isometric projection of the regulus and a curve of the system, $a=1$, $b=-2$, $c=1$, $d=2$, $p=4$, $\phi=45°$. Only one complete string, $A'AP$, has been drawn, and the part of the curve which is mechanically possible is indicated by its ordinates. In order to obtain the complete curve as represented by the equation the part shown must be reflected on the line s.

3. In subways with circular cross sections and lined with glazed tiling one may observe peculiar curves of reflexion emenating from the incandescent lights and winding on the surface in an ultimate direction towards the eye of the observer.*

To study the nature of these curves, assume in Fig. 2, a perspective view, L as a source of light near the cylindrical surface of the subway and B as the eye of the observer.

It is clear that every ray emanating from L that strikes the surface at P and is reflected to B is coplanar with the reflected ray PB. Furthermore, the perpendicular of incidence is a normal to the surface and consequently passes through the axis of the cylinder. Hence, in order to obtain points of reflexion P, pass a pencil of planes through the line connecting L and B. Each of these planes cuts the axis of the cylinder in a point A, and the surface normal from A, cutting LB at C, meets the surface in a point P of the curve containing reflecting points.

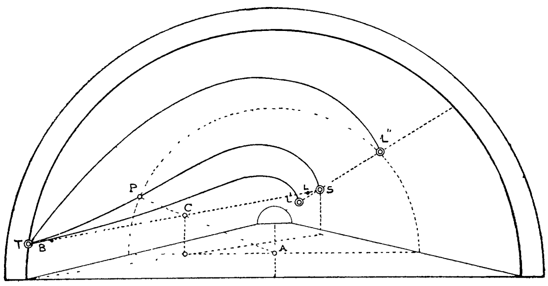

Fig. 2.

All normals PA from points of the curve pass through LB and the axis of the cylinder and are parallel to the planes normal to the axis; they form therefore an hyperbolic paraboloid. *The curve thus appears as the intersection of this regulus and the cylinder and is conseqnently of the fourth order.*

* Seen in the "Traforo," a subway under the Royal Gardens of the Quirinale in Rome.

It passes through the piercing points S and T, near L and B, of LB with the cylindrical surface. Every point of the cylinder gives rise to such a curve, so that a doubly infinite number of curves is obtained which all pass through T. In Fig. 2, three lamps at L, L', L'' at equal intervals and their corresponding curves are shown.

A number of problems suggest themselves in connection with a detailed study of these curves. As an example the result may be stated, that the locus of the foci of all conics (ellipses) cut out on the surface by the planes of the pencil through LB is a twisted quartic, whose orthogonal projection on a plane normal to LB passes through the circular points.

Referring again to Fig. 2, it is clear that the quartic through S and T is determined by ST and does not depend on the position of L and B on ST. Hence assuming on ST, L and B arbitrarily, there are generally only a limited number of points on the quartic where reflexion towards B takes place, so that geometrically there are no continuous curves of reflexion.

The reason why such curves are seen physically lies in the fact that in reality the tunnel-surface consists of rectangular plane pieces and the source of light of a luminous body. In place of an incident and reflected ray at each point of the quartic we therefore have bundles of rays striking the plane portions around each point of the curve. On account of the proximity of L and B to the surface, parts of each bundle will be reflected to B.

ON COMPOSITE NUMBERS P WHICH SATISFY THE FERMAT CONGRUENCE $a^{P-1} \equiv 1$ mod P.*

By R. D. CARMICHAEL, Indiana University.

Professor J. H. Jeans† has discussed the question of the converse of Fermat's theorem, showing that the relation

$$a^{P-1} \equiv 1 \text{ mod} P, \tag{1}$$

which (by Fermat's theorem) is always true when P is a prime for any value of a which is prime to P, is for any particular value of a true for values of P which are not prime. Mr. E. B. Escott‡ has given a more direct proof of the same theorem.

The failure of the converse of Fermat's theorem has also been pointed out by Lucas§ by means of the example

* Read before the American Mathematical Society, October 28, 1911.

† *Messenger of Mathematics*, 27 (1897-8), p. 174.

‡ *Messenger of Mathematics*, New Series, No. 431 (1907), p. 175.

§ *Theorie des nombres*, I, p. 422.

$$2^{37.73-1} \equiv 1 \bmod 37.73.$$

Lucas states the true converse in this form: *If $a^x—1$ is divisible by P for $x=P-1$, but for no other value of x which is a divisor of $P—1$, then P is a prime number.*

A detailed study of (1) for composite P has been made by Cipolla.* Among other things he shows that for every a there exists an infinite number of composite integers P satisfying (1). Conversely, for every odd P, not a power of 3, the congruence (1) is always satisfied by some number a different from ± 1.

The object of the present note is to extend the preceding results by proving the theorem that there are values of composite P for which relation (1) is true when a is any number prime to P. A necessary and sufficient condition for this will be given and a method will be explained for obtaining the appropriate values of P. The note is an amplification of an earlier remark by the present writer.†

Let a function $\lambda(m)$, where

$$m=2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n},$$

and the numbers $p_1, p_2, \dots, p_n$ are different odd primes, be defined in the following manner:

$$\lambda(2^{\alpha})=\phi(2^{\alpha}), \text{ if } \alpha=0, 1, 2;$$
$$\lambda(2^{\alpha})=\tfrac{1}{2}\phi(2^{\alpha}), \text{ if } \alpha>2;$$
$$\lambda(p_i^{\alpha_i})=\phi(p_i^{\alpha_i}), \text{ when } p_i \text{ is an odd prime};$$
$$\lambda(m)=\text{least common multiple of } \lambda(2^{\alpha}), \lambda(p_1^{\alpha_1}), \dots, \lambda(p_n^{\alpha_n}).$$

Then it is well known that for every a prime to P we have the congruence

$$a^{\lambda(P)} \equiv 1 \bmod P. \tag{2}$$

Furthermore, it has been proved‡ that $\lambda(P)$ is the least exponent such that (2) is true for every a prime to P. Hence, it follows at once that if (1) is true $\lambda(P)$ must be a factor of $P—1$. Again, if $\lambda(P)$ is a factor of $P—1$ the relation (1) is satisfied for every a prime to P. Hence the following theorem:

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* *Annali di Matematica* (3) 9 (1903), pp. 139-160.

† *Bulletin of the American Mathematical Society*, Vol. 16 (1910), pp. 237-238.

‡ *Bulletin of the American Mathematical Society*, Vol. 16 (1910), pp. 232-238.

THEOREM I. *A necessary and sufficient condition on the integer P in order that the congruence*

$$a^{P-1} \equiv 1 \bmod P$$

shall be true for every a which is prime to P is that $P-1$ shall be divisible by $\lambda(P)$; or

$$P-1 \equiv 0 \bmod \lambda(P). \tag{3}$$

From this theorem it follows at once that P and $\lambda(P)$ are relatively prime. Hence, P does not contain a repeated prime factor; for, if so, such a prime would be a factor both of P and of $\lambda(P)$—which we have just seen to be impossible. Moreover, P cannot be a product of two prime factors; for if $P=pq$ and $p>q$, it follows from theorem I that

$$\frac{pq-1}{p-1} = \text{integer}.$$

But

$$\frac{pq-1}{p-1} = q + \frac{q-1}{p-1}.$$

Since p is greater than q the second member of the last equation is not an integer. That is, $P=pq$ does not in any case satisfy theorem I. Now, $\lambda(P)$ is even since $\lambda(m)$ is even when $m \neq 2$ and P is composite so that it is not 2. Hence (3) cannot be satisfied by an even P. Collecting these results, we have

THEOREM II. *In order that composite P shall satisfy the congruence*

$$a^{P-1} \equiv 1 \bmod P$$

for every a which is prime to P it is necessary that a shall be the product of three or more different odd prime factors.

We shall now prove the following theorem:

THEOREM III. *There are values of composite P for which the congruence*

$$a^{P=1} \equiv 1 \bmod P$$

is true when a is any integer prime to P.

We shall prove this theorem by actually finding numbers P of the form

$$P=pqr$$

which satisfy the necessary and sufficient condition of theorem I, the numbers p, q, r being primes.

Evidently, a necessary condition that $P=pqr$ shall satisfy (3) is that each of the expressions

$$\frac{pqr-1}{p-1}, \quad \frac{pqr-1}{q-1}, \quad \frac{pqr-1}{p-1}$$

shall be an integer. Subtracting from these numbers in order the integers qr, rp, pq we have the result that the remainders

$$\frac{qr-1}{p-1}, \quad \frac{pr-1}{q-1}, \quad \frac{pq-1}{r-1} \tag{4}$$

must each be an integer. We shall refer to this as condition (4).

If $p=3$ it is easy to show that there is the single number 3.11.17 which satisfies condition (4), but that this number fails to satisfy the condition in theorem I. For $p=5$ two solutions satisfying (4) and theorem I are found; namely, 5.13.17 and 5.17.29. For $p=7$ we have the four solutions

7.13.19, 7.13.31, 7.19.67, 7.31.73.

We shall illustrate the method of finding solutions by carrying out the process in detail for the case $p=7$.

For this case the first number in (4) is

$$\frac{qr-1}{6}.$$

In order that this shall be an integer it is necessary and sufficient that both q and r shall be of the form $6n+1$ or both of the form $6n-1$. From the third number in (4) we see that

$$\frac{7q-1}{r-1}=m,$$

where m has one of the values 2, 3, ..., 6, since r is greater than q. This equation gives

$$(5) \qquad r=\frac{7q+m-1}{m}.$$

Substituting this value of r in the second member of (4) we find that we must have

$$\frac{4pq+6m-7}{m(q-1)}=\text{integer}=\frac{1}{m}\left(49+\frac{6m+42}{q-1}\right).$$

The values of q which satisfy this relation are the following:

For $m=2$, $q=19$;
For $m=3$, $q=13$, 31;
For $m=4$, $q=23$;
For $m=5$, $q=13$, 73;
For $m=6$, No value of q.

If we substitute in (5) and remember that r must be prime we see that $q=23$ and $q=73$ are both impossible. The other values of q in order give the numbers

7.19.67, 7.13.31, 7.31.73, 7.13.19

as the only possible values of P which are of the form $7qr$. Testing these values by means of theorem I, we see that each of them is a possible value of P as was stated above.

In a similar way we may assume other values of p and determine all possible values of P having such a prime factor p and having the desired property of satisfying (1) for every a prime to P. The modification of the method which is necessary for dealing with P as the product of four or more different odd primes is obvious. In this way one determines the following integers P* having the property that the congruence

* This list might be indefinitely extended.

$$a^{P-1} \equiv 1 \bmod P$$

is true for every a which is prime to P:

5.13.17	13.37.241
5.17.29	13.37.97
7.13.19	13.37.61
7.13.31	31.61.271
7.19.67	31.61.211
7.31.73	31.61.631
13.61.397	37.73.109

13.37.73.457

ON REMARKABLE POINTS OF CURVES.

By S. LEFSCHETZ, University of Nebraska.

Among the points that we define below as *remarkable*, the only ones that have been considered to any serious extent, are the well known singular points of algebraic curves. Plücker* in solving the famous Poncelet pardox on the class of the reciprocal of an algebraic curve, gave the formula connecting their numbers. Cailey† subsequently gave analogous ones for twisted curves, and Veronese‡ extended his results for n-space. In what follows an attempt is made to make precise the notion of remarkable point of a curve, by defining a class of points that can be reasonably so called, and two very simple propositions concerning these points are established. It is proper to remark that the discussion is not restricted to algebraic curves.

Let us consider in a plane an innumerable aggregate A of points, and the innumerable aggregate B of lines obtained by joining any two points of the aggregate A.

* *Algebraische Kurven.*

† *Coll. Math. Papers*, Vol. 1, p. 207.

‡ *Behandlung über die Methode des Projicirens und Schneidens, Math. Ann.* Vol. 19, p. 213.

Let C be a real curve given arbitrarily in the plane, subjected to the only condition that it shall have an analytic arc. Next consider the net of curves of order m passing through λ points of A and tangent to μ lines of B. For given values of m, λ, μ, it is clear that we have only an innumerable aggregate of these nets, since each net is a function of a finite number of points or lines taken from a discrete aggregate of them. In fact each net can be made to correspond to a series of terms depending upon $(\lambda+\mu)$ indices, and these evidently form an innumerable aggregate. Taking a definite net there is in general a number k such that at any point M of C there is a curve of the net having with C a contact of order k but none having with it a contact of order $(k+1)$. For some points of C, part of an innumerable aggregate, there will be at least one curve of the net having with C a contact of order $(k+1)$ at these points, and also some curves of the net will have with C a contact of order k in more than one point. It would be conceivable that all curves of the net having a contact of order k with C, satisfy this condition for h points, in which case we would single out those having a contact of order k in $(h+1)$ points. The points where there is a contact of order $(k+1)$ with a curve of the net, and the points where C is touched by the curves that have $(h+1)$ times a contact of order k with it will be called *remarkable points*, and we will include in this definition, the points of C corresponding to the remarkable points of its reciprocal with respect to an arbitrary conic of its plane. For given values of (m, λ, μ) it is clear that the points in question form a discrete aggregate on C, and therefore the same is true when m, λ, μ take all integral values. Hence, *the remarkable points of a curve form a discrete aggregate on it.*

Let O be an arbitrary point in the plane of C, and join to it all remarkable points of C, thus obtaining an innumerable aggregate of rays. As the lines going through O form an aggregate having the power of the continuum, if we subtract from it the above discrete aggregate of rays, the remaining one has still the power of the continuum.* Hence, *the aggregate of lines cutting a curve in none of its remarkable points has the power of the continuum.* This proposition would clearly be unmodified if instead of straight lines we took any algebraic curve.

The remarkable points as above defined include evidently a great variety of points seldom associated. Thus vertices of a conic are points of hyperosculation for the circles of curvature of the curve. There the aggregate A is formed of the two circular points at infinity. A multiple point is the reciprocal of points of contact of a multiple tangent of the reciprocal polar of C, and therefore is also included. In this case $A=B=0$; a multiple point is a remarkable point independent of the aggregate A.

Finally, we may say that the extension to n-space varieties is quite evident.

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* E. Borel, *Lecons sur la Theorie des Fonctions*, p. 13.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

360. Proposed by CHARLES C. GROVE, Columbia University, New York.

A bridge club of 28 members has 27 meetings. There are 7 tables with 4 members at each table. Can the players be so arranged that at the end of the season (27 meetings) each member will have played *with* every other member one game and *against* every other member two games, one game meaning one meeting; and how?

Solution by the PROPOSER.

The number of combinations of 28 people two at a time, ${}^{28}C_2$, is 14 times 27; of which each individual appears in ${}^{27}C_1$ or 27.

Thus each member plays *once with* each other member. At each of the 27 meetings 14 combinations of two each are playing, and so in the season all the possible combinations will be used, 14×27. Further, at each meeting any member is playing *with* one member and *against* two other members. Since he thus goes around the membership *once with* each he will be going around *twice against each.*

The method of arranging the players can be seen from the following:

1, 2	2, 3	3, 4	4, 5	...	26, 27	27, 28
1, 3	2, 4	3, 5	4, 6	...	26, 28	
1, 4	2, 5	3, 6	¦			
1, 5	¦	¦	4, 28			
1, ¦	¦	3, 28				
¦	2, 28					
1, 28						

where the numbers represent the different members of the club.

At the various meetings the tables must be filled from the above combinations.

Meeting	Table I.	II.	III.	IV.	V.	VI.	VII.
	1	5	9				
First	3 4	7 8	11 12	. . . .			
	2	6	10				
	1	5	9				
Second	2 4	6 8	10 12	. . . .			
	3	7	11				

	1	5	9	
Third	2 3	6 7	10 11	. . . .
	4	8	12	

All the combinations involving 1, 2, 3, 4 only have been used, and thereby 1 has played *with* each 2, 3, 4 once and twice against each. Since the players are identically involved, *mutatis mutandis,* all the other players will have a similar experience.

361. Proposed by C. E. GITHENS, Ph. D., Wheeling, W. Va.

Find three integral values for $[-10+9\sqrt{(-3)}]^{1/3}+[-10-9\sqrt{(-3)}]^{1/3}$. A solution not involving a cubic is desired.

I. Solution by the PROPOSER.

I. Put $(-10+9\sqrt{-3})^{\frac{1}{3}}+(-10-9\sqrt{-3})^{1/3}=\frac{1}{2}[(-80+72\sqrt{-3})^{1/3}+(-80-72\sqrt{-3})^{1/3}]$.

Let $(-80+72\sqrt{-3})^{1/3}=\sqrt{x}+\sqrt{y}$ and $(-80-72\sqrt{-3})^{1/3}=\sqrt{x}-\sqrt{y}$.

Then, $[6400-(-15552)]^{1/3}=x-y=28$, and, therefore,

$(-80+72\sqrt{-3})^{1/3}=\sqrt{(x+28)}+\sqrt{x}$. Hence, by raising both sides to the third power, $(4x+28)\sqrt{(x+28)}+(4x+84)\sqrt{x}=-80+72\sqrt{-3}$.

Put $4x+28\sqrt{(x+28)}=-80$, and $4x+84\sqrt{x}=72\sqrt{(-3)}$.

Let $4x+28=-80$ or factor of -80; -40, -20, -16, -10, -9, -8, -5, -4, -2, and

$4x+84=72\sqrt{(-3)}$ or factor of 72; 36, 24, 18, 12, 9, 8, 6, 4, 3, 2.

Subtracting, $-56=-20-(36)$.

$\therefore 4x+28=-20$ and $x=-12$.

[1] $\frac{1}{2}[(-80+72\sqrt{-3})^{1/3}+(-80-72\sqrt{-3})^{\frac{1}{3}}]$

$=\frac{1}{2}[(\sqrt{(x+28)}+\sqrt{x})+(\sqrt{(x+28)}-\sqrt{x})]=4$, answer.

II. Similarly with $(-80+72\sqrt{-3})^{\frac{1}{3}}=\sqrt{(x_0+28)}-\sqrt{x_0}$, in which the factors -80 and 24 added to eliminate the $4x$'s produces

$x_0=-27$ and $\frac{1}{2}[(\sqrt{(x_0+28)}-\sqrt{x_0})+\sqrt{(x_0+28)}+\sqrt{x_0}]=-1$, answer.

III. With $-\sqrt{(x_1+28)}+\sqrt{x_1}$ and -16 and 72 as factors as above, $x=-3$ and $\frac{1}{2}[-\sqrt{(x_1+28)}+\sqrt{x_1}+(-\sqrt{(x_1+28)}-\sqrt{x_1})]^{\frac{1}{3}}=-5$, answer.

IV. The root $-\sqrt{(x_2+28)}-x_2$ produces no two factors of -80 and 72 whose difference or sum equals ± 56; hence it is not a root, which is as it should be, for the numerical equation is an example of the "irreducible case" in the Cardan solution of a cubic whose equation is $x^3-21x+20=0$.

II. **Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.**

Putting $(-10+9\sqrt{3}.\sqrt{-1})=\rho(\cos\phi+\sin\phi.\sqrt{-1})$, we get $\tan\phi=-\frac{9}{10}\sqrt{3}$. $\rho=\sqrt{343}$, and $(-10+9\sqrt{3}.\sqrt{-1})^{\frac{1}{3}}+(-10-9\sqrt{3}.\sqrt{-1})^{\frac{1}{3}}=2\sqrt{7}\cos\frac{1}{3}\phi$.

$\therefore 4\cos^3\frac{1}{3}\phi-3\cos\phi+\frac{10}{49}\sqrt{7}=0$.

By trial, $\cos\frac{1}{3}\phi=\frac{2}{7}\sqrt{7}$, and dividing the last trinomial by $\cos\frac{1}{3}\phi-\frac{2}{7}\sqrt{7}$, we get $4\cos^2\frac{1}{3}\phi+\frac{8}{7}\sqrt{7}.\cos\frac{1}{3}\phi-\frac{5}{7}=0$; whence $\cos\frac{1}{3}\phi=-\frac{5}{14}\sqrt{7}$, $\cos\frac{1}{3}\phi=\frac{1}{14}\sqrt{7}$.

$\therefore$ The three values required are 4, -5, 1.

Also solved by **A. M. Harding.**

GEOMETRY.

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386. Proposed by DANIEL KRETH, Oxford, Iowa.

Construct the triangle, having given, the vertical angle, the sum of the three sides, and the perpendicular.

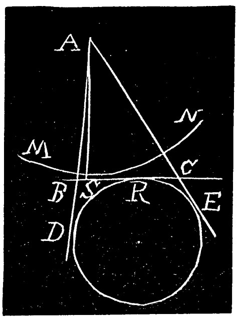
I. **Solution by H. PRIME, Boston, Massachusetts.**

Let ABC be the required triangle, C the given angle. On AB produced take $BE=BC$. On BA produced take $AF=AC$. Let O be the center of circle ECF. Then we have the angles $FOE=2(BEC+AFC)=ABC+BAC=$supplement of C.

Hence, to construct the triangle, on $EF=$the given sum of the three sides form the isosceles triangle EOF, making $EOF=$the supplement of the given vertex angle (or $OEF=OFE=$one half the given angle). About O as center describe the arc EF. Parallel to EF and at a distance from it equal to the given altitude draw a line meeting the arc at C and C'. Draw CA and CB, making the angles $ACF=AEC$ and $BCE=BEC$. ABC is the required triangle.

II. **Solution by C. N. SCHMALL, New York City, and A. M. HARDING, University of Arkansas.**

Construct an angle A equal to the given vertical angle. Lay off AD and AE each equal to *half* the given sum of the sides. Describe a circle touching these lines in D and E. With A as center and radius equal to the given perpendicular, describe a circle MN. By a well known method draw a line tangent to *both* these circles touching in R and S, respectively, and cutting the sides in B and C. Then ABC is the triangle required.



Proof. $BR=BD$, $CR=CE$.

$\therefore BC=BD+CE$; hence the triangle has the given perimeter. Also, AS is perpendicular to BC; therefore the triangle has the required altitude. Q. E. D.

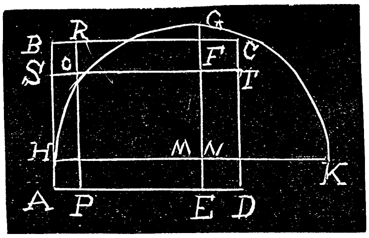
Also solved by **J. Scheffer** and **A. H. Holmes.**

387. Proposed by DANIEL KRETH, Oxford, Iowa.

A lot 100 feet long and 60 feet wide, has a walk extending from one corner half way around it, and occupying one-third of the area. Required the width of the walk. A geometrical construction is desired.

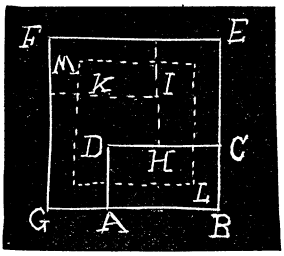
I. Solution by A. H. HOLMES, Brunswick, Maine.

Let $ABCD$ be the lot. $AB=CD=60$, and $AD=BC=100$. On DA take DE=one third of CD, and draw EF parallel to CD and cutting BC in F. On EF take EN=one fourth of DE and draw HN parallel to AD, the point H being on AB and $AH=EN$. On HN extended take $NK=NF$. Bisect HK in M, and with MH as radius describe a semi-circle on HK. Extend EF to cut the circumference in G. On EA take EP $=NG$. Draw PR parallel to AB. Take $BS=AP$, and draw ST parallel to BC. Then on ST take $SO=AP$. Then $ABRP+ORCT=\frac{1}{3}ABCD$, and AP =width of walk required; which is shown as follows: Let x=width of walk. Then $100x+(60-x)x=2000$. Therefore, $(80-x)^2=55\times 80$.



II. Solution by C. E. GITHENS, Ph. D., Wheeling, West Virginia.

Let $ABCD$ be the given lot. Form the square $GBEF$ by arranging three other equal lots as in the figure. Then $GB=60$ feet+100 feet =160 feet.



Area of square $DHIK=40^2$ square feet

Area of square $LM=\frac{2}{3}(4ABCD)+DHIK=$ 17,600 square feet.

Hence, side of square $LM=\sqrt{(17,600)}$ feet $=40\sqrt{11}$ feet.

Hence, width of walk$=\frac{1}{2}(160-40\sqrt{11})$ feet$=(80-20\sqrt{11})$ feet.

Also solved by H. Prime and S. G. Barton.

388. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics and Astronomy, Ohio University Athens, Ohio.

A conic is inscribed in a triangle and one focus lies on the polar circle of the triangle. Prove that the corresponding directrix passes through the center of perpendiculars.

Solution by the PROPOSER.

Reciprocating with respect to the focus, the conic corresponds to the circumscribing circle of the reciprocal triangle; the polar circle, whose center is the orthocenter of the fixed triangle, to a parabola with focus, the fixed focus of the given conic, the given orthocenter to the directrix of the

reciprocal parabola, the directrix of the conic to the center of the reciprocal circle which is on the directrix of the parabola.

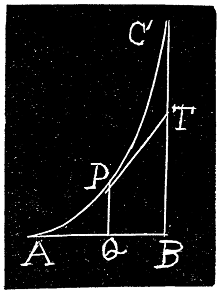
CALCULUS.

314. **Proposed by REV. J. H. MEYER, S. J., New Orleans, La.**

A fox started from a certain point and ran due east 300 yards, when it was overtaken by a hound that started from a point 100 yards due north of the fox's starting point, and ran directly towards the fox throughout the race. Find the lengh of the curve described by hound, both having started at the same instant, with a uniform velocity.

Solution by J. SCHEFFER, A. M., Hagerstown, Maryland, and FRANCIS E. RUST, E. E., Pittsburg, Pa.

Let A be the starting point of the hound, and B that of the fox, C the point of capture, P some point of the curve described by the hound, $AQ=x$, $PQ=y$, DPT a tangent to the curve at P, $AB=a$ $(=100)$, $BC=b(=300)$; $AP=s$, m=rate of hound, n= rate of fox; $s=mt$, $BT=nt$, t being a certain time. $BT=y+(a-x)\tan TBE=y+(a-x)\dfrac{dy}{dx}$.



$\therefore \dfrac{s}{m}=\dfrac{y+(a-x)(dy/dx)}{n}$; or, putting $\dfrac{n}{m}=\beta$, $\beta s=y+(a-x)\dfrac{dy}{dx}$. Differentiating, $\beta\dfrac{ds}{dx}=(a-x)\dfrac{d^2y}{dx^2}$; or

$\beta\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}=(a-x)\dfrac{d^2y}{dx^2}$. Putting $\dfrac{dy}{dx}=p$, $\beta\sqrt{(1+p^2)}=(a-x)\dfrac{dp}{dx}$.

$\therefore \beta\dfrac{dx}{a-x}=\dfrac{dp}{\sqrt{(1+p^2)}}$. $\therefore \dfrac{C_1}{(a-x)^\beta}=p+\sqrt{(1+p^2)}$, and since for $x=0$, $p=0$, $\dfrac{C_1}{a^\beta}=1$, and $C_1=a^\beta$. Now, $\dfrac{a^\beta}{(a-x)^\beta}=p+\sqrt{(1+p^2)}$, or $a^\beta(a-x)^{-\beta}=p+\sqrt{(1+p^2)}$; whence $p=\dfrac{dy}{dx}=\frac{1}{2}[a^\beta(a-x)^{-\beta}-a^{-\beta}(a-x)^\beta]$.

$\therefore y=\dfrac{a^{-\beta}(a-x)^{1+\beta}}{1+\beta}-\dfrac{a^\beta(a-x)^{1-\beta}}{1-\beta}+C_2$; but $0=\frac{1}{2}\left(\dfrac{a^{-\beta}a^{1+\beta}}{1+\beta}-\dfrac{a^\beta a^{1-\beta}}{1-\beta}\right)+C_2$, whence $C_2=\dfrac{a^\beta}{1-\beta^2}$.

$\therefore y=\frac{1}{2}\left[\dfrac{a^{-\beta}(a-x)^{1+\beta}}{1+\beta}-\dfrac{a^\beta(a-x)^{1-\beta}}{1-\beta}\right]+\dfrac{a^\beta}{1-\beta^2}$; $\dfrac{ds}{dx}=\frac{1}{2}[a^\beta(a-x)^{-\beta}+a^{-\beta}(a-x)^\beta]$;

$$\therefore s=-\tfrac{1}{2}\left[\frac{a^{\beta}\,(a-x)^{1-\beta}}{1-\beta}+\frac{a^{-\beta}\,(a-x)^{1+\beta}}{1+\beta}\right]+C_3.\quad \text{For } x=0,\ s=0.$$

$$\therefore C_3=\frac{a}{1-\beta^2}.\quad \therefore s=\frac{a}{1-\beta^2}-\tfrac{1}{2}\left[\frac{a^{\beta}\,(a-x)^{1-\beta}}{1-\beta}+\frac{a^{-\beta}\,(a-x)^{1+\beta}}{1+\beta}\right].$$

Since for $x=a$, $y=b$, we have $b=\dfrac{a\beta}{1-\beta^2}$; or, substituting $a=100$, $b=300$, we have $\beta^2+\frac{1}{3}\beta=1$; whence $\beta=\frac{1}{6}(\sqrt{37}-1)$ and length of curve between A and $C=\dfrac{a}{1-\beta^2}=50(\sqrt{37}+1)=354.135$.

$$y=\frac{na}{n^2-1}-\frac{na^{1/n}}{2(n-1)}(a-x)^{(n-1)/n}+\frac{n}{2(n+1)}\,\frac{1}{a^n}(a-x)^{(n-1)/n}.$$

315. Proposed by C. N. SCHMALL, New York City.

If $y=f(x)$, show by Taylor's Theorem that

$$f\left(\frac{x}{1+x}\right)=y-\frac{x^2}{1+x}\cdot\frac{dy}{dx}+\frac{x^4}{2\,(1+x)^2}\cdot\frac{d^2y}{dx^2}-\frac{x^6}{2.3.\,(1+x)^3}\cdot\frac{d^3y}{dx^3}+\dots\text{ etc.}$$

Solution by the PROPOSER.

Put $\dfrac{x}{1+x}=x+h$, then $f\left(\dfrac{x}{1+x}\right)=f(x+h)$,

$$\text{and also, } h=\frac{x}{1+x}-x=-\frac{x^2}{1+x}.$$

$$\therefore h^2=\frac{x^4}{(1+x)^2},\quad h^3=-\frac{x^6}{(1+x)^3},\ \text{and so on.}$$

Substituting these values of the powers of h in Taylor's series, we have the required result; *i. e.*, from

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\text{etc.},$$

we get, by direct substitution,

$$f\left(\frac{x}{1+x}\right)=y-\frac{x^2}{1+x}\,\frac{dy}{dx}+\frac{x^4}{2\,(1+x)^2}\,\frac{d^2y}{dx^2}-\frac{x^6}{2.3(1+x)^3}\,\frac{d^3y}{dx^3}+\text{etc.}$$

316. Proposed by C. N. SCHMALL, New York City.

$$\int_0^{\infty}\frac{\cos ax}{1+x^2}dx=\tfrac{1}{2}\pi\,e^{-a}=\int_0^{\infty}\frac{x\sin ax}{1+x^2}dx.$$

(From Bromwich, *Theory of Infinite Series*, p. 442, ex. 5, and also from Carslaw, *Fourier's Series*, p. 113, ex. 12.) Prove this by any method.

Solution by A. M. HARDING, Fayetteville, Arkansas.

It can easily be shown that

$$\int_0^\infty e^{-bx}\cos bm\,db=\frac{x}{m^2+x^2};\quad \int_0^\infty e^{-bx}\sin bm\,db=\frac{m}{x^2+m^2}$$

$$\text{and } 2\int_0^\infty be^{-b^2(1+x^2)}db=\frac{1}{1+x^2}.$$

Hence $u=\int_0^\infty \frac{x\sin ax}{1+x^2}dx=\int_0^\infty \sin ax\left(\int_0^\infty e^{-bx}\cos b\,db\right)dx.$

$$\therefore \frac{du}{db}=\int_0^\infty \sin ax e^{-bx}\cos b\,dx=\cos b\int_0^\infty e^{-bx}\sin ax\,dx=\cos b.\frac{a}{a^2+b^2}.$$

$\therefore u=\int_0^\infty \frac{a\cos b}{a^2+b^2}db$, putting ax for b, $=\int_0^\infty \frac{\cos ax}{1+x^2}dx.$

$$\therefore \int_0^\infty \frac{x\sin ax}{1+x^2}dx=\int_0^\infty \frac{\cos ax}{1+x^2}dx.$$

Now $u=\int_0^\infty \frac{\cos ax}{1+x^2}dx=\int_0^\infty \cos ax\left(2\int_0^\infty be^{-b^2(1+x^2)}db\right)dx.$

$$\frac{du}{db}=2\int_0^\infty \cos ax.be^{-b^2(1+x^2)}dx=2be^{-b^2}\int_0^\infty e^{-b^2x^2}\cos ax\,dx=2be^{-b^2}.\frac{\sqrt{\pi}}{2b}e^{-(a^2/4b^2)}$$

(see Byerly's *Integral Calculus,* Art. 93 (b))

$=\sqrt{(\pi)}e^{-b^2-[(a/2)^2/b^2]}.\quad u=\sqrt{(\pi)}\int_0^\infty e^{-b^2-[(a/2)^2/b^2]}db=\sqrt{(\pi)}.e^{-a}.\frac{\sqrt{\pi}}{2}=\frac{\pi}{2}e^{-a}.$

(Art. 93 (a)).

Also solved by J. Scheffer, S. A. Corey, and the Proposer.

NOTES AND NEWS.

Professor Cajori's *Theory of Equations,* which was published in 1904, was reprinted by the Macmillan Company during last January. M.

The spring meeting of the Chicago section of the American Mathematical Society will be held at Chicago on Friday and Saturday, April 5 and 6, 1912. S.

The third number of the *Tohoku Mathematical Journal*, published at Sendai, Japan, appeared during last January. It contains seven articles, of which six are in English while the remaining one is in German. A volume composed of four numbers, is sold for one dollar. M.

The Department of Superintendence of the National Education Association held its annual meeting in St. Louis, on February 27, 28, and 29, 1912. Many interesting questions are found on its program and on those of the various educational societies which met in St. Louis during the same week. S.

The remaining two volumes of the second edition of Pascal's *Repertorium der hoeheren Mathematik* are now expected to appear toward the end of the present year. It may be remembered that three successive circulars issued by B. G. Tuebner, of Leipzig, announced that these volumes would appear in the spring and summer of 1911 and at Easter of 1912. M.

During last September the University of Illinois published a list of serials in the university library. This includes newspapers, magazines, periodicals, and serial publications of societies, corporations, institutions, and governmental bodies. These serials have appeared under more than 8000 different titles and 215 of them are classified under mathematics, astronomy, or physics. M.

In an interesting article on non-euclidean geometry in the current number of the *Bulletin of the American Mathematical Society*, it is found by actual count that of all who have ever written on non-euclidean geometry, Dr. Halsted has the most titles to his credit, seventy-seven with his translations, which with his biography of Sommerville bring the count up to ninety-two. F.

The North Central Association of Schools and Colleges will hold its annual meeting at the Auditorium Hotel in Chicago on Friday and Saturday, March 22 and 23, 1912. An important report on the accrediting of schools will be presented by a committee of which President Hill, of the University of Missouri, and Director Judd, of the University of Chicago School of Education, are members. S.

The annual conference of the University of Chicago with its coöperating schools, which for twenty years has been held in November, will this year be held in April. The date was changed in order to give the representatives of secondary schools time to visit the junior college work of the university which they have been doing all the autumn and winter. The programme of the conference will be based upon the reports of their visitations from the schools to the university, thus reversing the usual order of procedure. S.

At the University of Illinois the following mathematical courses, beyond a first course in calculus, are now being given: Functions of a Complex variable, Professor Townsend; Theory of Numbers and Theory of Equations, Professor Miller; Theory of Statistics, Professor Rietz; Solid Analytic Geometry, Professor Sisam; Theory of Potential, Professor Shaw; Projective Geometry and Linear Transformations, Professor Emch; Functions of a Real Variable, Dr. Crathorne; Partial Differential Equations, Dr. Wahlin; Advanced Calculus, Dr. Lytle. M.

The Edinburgh Mathematical Society is publishing a brief review of elementary mathematics and science, entitled *Mathematical Notes.* The series began in April, 1909, and the nine numbers which have appeared cover 108 pages in all. No. 9 was issued in January, 1912, and contains four short papers entitled respectively: "Logarithms and the Reciprocals of Numbers; Certain Processes in the Theory of Equations Illustrated Geometrically; The Arithmetic Mean of a Number of Real Positive Numbers is not Less than Their Geometric Mean; Additional Notes on the Right-Angled Triangle. All contributions and communications referring to these notes should be addressed to P. Pinkerton, George Watson's College, Edinburgh, Scotland." M.

The sixth number of the journal published by the new Spanish Mathematical Society under the title *Revista de la Sociedad Matematica Espanola* appeared during last February. In addition to the usual articles this number contains a list of the 423 members of this young and active society. From this list it appears that the Spanish speaking countries of America are, as yet, too poorly represented in this society. As the articles appearing in the "Revista" are quite elementary and represent a number of different interests, it is to be hoped that this journal will serve to awaken a more general mathematical interest in the countries which employ the Spanish language, and that it may soon find more hearty support from this side of the Atlantic. M.

The provisional report of the national committee of fifteen on Geometry Syllabus, which was presented at the San Francisco meeting of the National Education Association, in July, 1911, and also of the meeting of the American Federation of the Teachers of the Mathematical and Physical Sciences, held in Washington in December, 1911, has since been revised by the committee in the light of many helpful suggestions received from many sources, and a large edition has been issued for general distribution. Copies will soon be mailed to those who are members of the associations belonging to the Federation. Others who desire copies may secure them *gratis* upon application to the Commissioner of Education, Department of the Interior, Washington, D. C., to whom a large number has been donated for general distribution. S.

In connection with the large French Mathematical Encyclopedia there is being published a "Tribune Publique" devoted to corrections and additions relating to the parts of this great work which have been issued. The entire mathematical world is invited to help in this way toward making this work more complete and more reliable. In 1904 G. Enestroem called attention, in the Bibliotheca Mathematica, page 398, to the usefulness of such general coöperation and to the fact that it is important to omit reference to useless or unreliable articles, as well as to give reference to all articles, which advance the subject in hand. It is to be hoped that the "Tribune publique" will receive more and more general support and that an increasing number of mathematicians will realize the dignity and importance of assisting, even in this humble manner, in this useful but colossal enterprise.
M.

From the 9th to the 15th of October, 1911, vacation courses were offered at Zurich, Switzerland, for the benefit of all the secondary teachers of Switzerland. There were 520 in attendance and forty-eight different courses were given. The following is a list of the mathematical courses, according to the January, 1912, number of L'Ensignement Mathématique: Introduction to the theory of groups, 6 hours; Astronomical observations and determinations of a position, 3 hours; The foundations of geometry, 5 hours; Vectorial analysis, 4 hours. The following three subjects were discussed: The notion of function in secondary instruction; Agreement between technical drawing and descriptive geometry; The use of certain problems of physics as applications in the teaching of mathematics. Some of these courses and subjects for discussion appear to be of a higher grade than those usually demanded by our teachers of secondary mathematics. M.

B. G. Teubner of Liepzig, Germany, has begun the publication of a series of small volumes at 20 cents each, which are intended to give an elementary exposition of various parts of elementary mathematics, and its contact with more advanced subjects. According to a recent announcement (January, 1912), the following four volumes have appeared: Numerals and number systems of ancient and modern civilized countries; The concept of number with its logical and historical significance; The Pythagorean theorem and its bearing on Fermat's theorem; and Calculus of probability with applications. The general series bears the name "Mathematische Bibliothek" and is edited by Dr. W. Leitzmann and Dr. A. Witting. It aims to enable those who are interested in mathematics in the widest sense to pursue the subject beyond what is ordinarily presented in the schools. The fact that these volumes can be sold at such a small price seems to indicate that such literature finds a large number of buyers in Germany and directs attention to the extravagant price charged in this country even for elementary text-books.
M.

BOOKS.

Tables of Physical and Chemical Constants and Some Mathematical Constants. By G. W. C. Kaye, B. A. (Cantab.), D. Sc. (Lond.), A. R. C. Sc. (Lond.), The National Physical Laboratory; late Sub-Lector in Physics, Trinity College, Cambridge, and T. H. Laby, B. A. (Cantab.), Professor of Physics, Wellington, N. Z.; formerly Exhibition of 1851 Scholar; Joule Student; and Research Exhibitioner, Emanuel College, Cambridge. Large 8vo. Flexible cloth back, vi+153 pages. Price, $1.50. New York: Longmans, Green & Co.

A desideratum of every present-day physics teacher has been the satisfying of the need of an accurate, authorative, and inexpensive set of physical and chemical tables for the laboratory. The book before us satisfies this need admirably. In this book, a half dozen pages are devoted to the various units,—absolute and derived. Then follows tables dealing, among other things, with terrestrial and astronomical constants, barometry, hydrometry, and densities, screws and wire gauges, elasticities and tensil strengths, viscosities, molecular constants and kinetic theory, critical data, diffusion, surface tensions, hygromety, and vapor pressure.

In the pages on heat, considerable attention is given to present-day thermometry, much data concerning the melting and boiling point, standard temperatures, expansive coefficients, thermal conduction, the recent determinations of Joule's mechanical equivalent, specific and latent heats, black-body radiation, the solar constants, etc. The more recently determinant value for latent heat of steam is given, viz., 540.

In the same way, we find here the various constants in sound, light, electricity, magnetism, and radio-activity. About fifteen pages are devoted to the physical constants of chemical compounds. The book also contains 4-figure logs and anti-logs, 5-figure logs $e-x$ and the more usual trigonometric functions with reciprocals and squares.

No physical nor chemical laboratory should be without these tables. They are absolutely indespensible to the wide-awake physics and chemistry teacher. F.

Plane Geometry. By C. A. Hart, Instructor in Mathematics, Wadleigh High School, New York City, and Daniel D. Feldman, head of the Department of Mathematics, Erasmas Hall High School, Brooklyn, N. Y., with the Editorial Coöperation of J. H. Tanner and Vergil Snyder, Professor of Mathematics in Cornell University. 8vo. Cloth, vii+303 pages. New York: American Book Co.

In addition to the usual theorems, demonstrations, and a large collection of exercises, human interest is added by a number of pictures of leading Geometricians, accompanied by brief biographical sketches. It will prove to be a very useful book in the hands of an enthusiastic teacher. F.

Practical Algebra, Second Course. By Joseph V. Collins, Ph. D., Professor of Mathematics, State Normal School, Stevens Point, Wisconsin. 8vo. Cloth, viii+303 pages. New York: American Book Co.

In the preparation of this second course, the author had the benefit of suggestions and criticisms from a number of teachers whose practical experience in the school room enable them to give valuable aid in the production of a second book in algebra.

Many of the features of the author's First Year Course have been retained in the Second Course. The book is well written and the presentation of the principles of algebra is clearly and forcefully put before the student's mind. F.

Elementary Plane Geometry. By John C. Stone, A. M., Head of the Department of Mathematics, State Normal School, Montclair, N. J., Co-Author of the Southworth-Stone Arithmetics, Stone-Millis Secondary Arithmetics, Algebras, etc., and James F. Millis, A. M., Head of the Department of Mathematics, Francis Parker School, Chicago, Co-Author of the Stone-Millis Secondary Arithmetics and Algebras. 8vo. Red cloth sides, viii+252 pages.

In this text, much is made of every-day practical problems as examples and exercises coming under application of theorems. Also the trigonometric functions are defined and used to a very limited extent.

On page 226, Dürer's construction of a pentagon appears as exercise 20. A figure is drawn and the student is asked to explain the construction and prove that this figure is a *regular* pentagon. We wonder how the authors expect this to be done. F.

Engineering Applications of Higher Mathematics. By V. Karapetoff, Part I. Problems on Machine Design. First Edition, first thousand. 8vo. xiv+69 pages. Price, 75 cents. New York: John Wiley & Sons.

The aim of this book is not intended to present the principles of the calculus to the student, but, rather, the author tells us, to enable an engineer to make better use of his higher mathematics in his work. To this end, he has taken up a brief treatment of loads on the inclined plane, friction in journals, friction in step bearings, carrying capacity in belts, torsion of shafts, and moments of inertia of flywheels. Under each head a number of problems are solved in detail. The book will be found helpful to the practical engineer who has limited time for more extensive study. F.

Elements of Applied Mathematics. By Herbert E. Cobb, Professor of Mathematics in Lewis Institute, Chicago. 12mo. Cloth, 274 pages. Price, $1.00. Chicago: Ginn & Co.

"The idea that mathematics is a series of discrete subjects, each in turn to be studied and dropped without reference to the others, or to the mathematical problems to be met in the future, is fast being displaced by that which links arithmetic, algebra, geometry, and trigonometry in close relationship with each other, and connects all our mathematics with our work in the shops and laboratories.

Elements of Applied Mathematics is constructed upon this latter principle. The work outlined consists largely of lists of problems based on the student's preceding work in mathematics, illustrating the work in the shops and laboratories, and of simple experiments and exercises in the mathematics classroom, where the pupil, by measuring and weighing, secures his own data for numerical computations and geometrical constructions."

First Year Algebra. By William J. Milne, Ph. D., LL. D., President, New York State Normal College, Albany, N. Y. 8vo. Cloth, 320 pages. New York: The American Book Co.

This is a very excellent book for the beginner, stimulating and instructive. F.

Brief Course in Analytical Geometry. By J. H. Tanner, Professor of Mathematics, Cornell University, and Joseph Allen, Assistant Professor of Mathematics in the College of the City of New York. 12mo. Cloth and leather back, x+282+xxiv pages. New York and Chicago: The American Book Co.

This little book preserves the main features of the authors' Elementary Analytical Geometry, a book well adapted to elementary instruction in colleges. Those who have used Elementary Analytical Geometry and found it a little too comprehensive for the time at their disposal will want to try this briefer course by the same authors. F.

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ON THE SUM OF THE NUMBERS WHICH BELONG TO A FIXED EXPONENT AS REGARDS A GIVEN MODULUS.

By G. A. MILLER, University of Illinois.

§ 1. Introduction.

In article 81 of Gauss's *Disquisitiones Arithmeticae*, 1801, it is proved that the sum of the incongruent primitive roots of a prime number p is $\equiv 0 \pmod{p}$ whenever $p-1$ is divisible by the square of a prime; when $p-1$ is not divisible by the square of a prime, this sum is $\equiv 1$ or $\equiv -1$ according as $p-1$ is the product of an even or of an odd number of distinct primes. As this theorem can be so readily verified it is within easy reach of those whose mathematical attainments are very meagre. We proceed to illustrate it by using successively for p the numbers 13, 11, and 31.

The four primitive roots of 13 are 2, 6, 7, 11. Their sum is $26 \equiv 0$ (mod 13) and 12 is divisible by 2^2. The four primitive roots of 11 are 2, 6, 7, 8. Their sum is $23 \equiv 1$ (mod 11) and 10 is the product of two distinct primes. The eight primitive roots of 31 are 3, 11, 12, 13, 17, 21, 22, 24. Their sum is $123 \equiv -1$ (mod 31) and 30 is the product of an odd number of distinct primes.

The given theorem, due to Gauss, has been proved in various ways and was extendod by Arndt in 1846.* In Bachmann's *Niedere Zohlentheorie*, 1902, page 333, we find the following much more general theorem: The sum of the incongruent numbers which belong, with respect to mod p^a or mod $2p^a$ (p being any odd prime), to any exponent which is divisible by the square of a prime is always $\equiv 0$; when this exponent is not divisible by the square of a prime, the given sum is $\equiv 1$ or $\equiv -1$ according as the exponent is the product of an even or of an odd number of distinct primes.

As It is so very easy to verify that this general theorem is not universally true it is singular that it should have appeared in such an excellent work, even if it is corrected in the Zusatze at the end of the volume. For instance, if we let $p \equiv 3$ and $\alpha \equiv 3$, it is evident that 10 and 19 are the two incongruent numbers which belong to exponent 3. Their sum is $29 \equiv 2$ (mod 27)

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instead of being $\equiv -1$ in accord with the general theorem. On the other hand, the six incongruent numbers which belong to exponent 9 (mod 27) are as follows: 4, 7, 13, 16, 22, 25. Their sum is $87 \equiv 6$ instead of being $\equiv 0$ in accord with this theorem.

In the present paper we aim to give a few new theorems relating to the sum of the numbers which belong to a given exponent, to give a simple proof of the generalization of Gauss's theorem due to Arndt, and also to exhibit clearly some relations between number theory and group theory which are connected with particular developments. Various points of contact between these two theories have been noted before, but their common ground doubtless offers still much to be investigated. It may be remembered that Poincaré called particular attention to the fact that the borderlands between various mathematical fields give the greatest promise for important advances.*

§ 2. Theorems Relating to a General Modulus.

It is well known that the $\phi(m)$ positive integers which do not exceed the positive integer m and are prime to it constitute an abelian group G with respect to multiplication (mod m).† These $\phi(m)$ numbers are said to constitute a reduced system of residues (mod m), and it is evident that 1 and $m-1$ belong to exponents 1 and 2 respectively. Little is known as regards the exponents of the other numbers except that each of the exponents is a divisor of $\phi(m)$.

Since $m-1 \equiv -1$ (mod m) corresponds to an operator of order 2 in G, and the products of all the operators of G by any one of them give each of these operators once and only once, it results that we obtain a reduced system of residues (mod m) by multiplying each number of such system by -1. In particular, the sum of all the numbers of any reduced system of residues (mod m) is $\equiv 0$, since this sum is not altered when we change its sign. If we multiply an operator whose order is divisible by 4 by any operator of order 2 which is commutative with it the order of this product is the same as the order of the given operator. Hence it results directly that *the sum of all the numbers of any reduced system of residues (mod m), which belong to any exponent which is divisible by 4 must always be $\equiv 0$ (mod m).*

To illustrate this theorem we may consider the reduced system of residues mod 20. The eight numbers of this reduced system are evidently as follows:

$$1,\ 3,\ 7,\ 9,\ 11,\ 13,\ 17,\ 19.$$

The four numbers which belong to exponent 4 are 3, 7, 13, 17. Their sum is clearly $\equiv 0$ (mod 20). It may be observed that the sum of the three

*Poincaré, *Bulletin des Sciences Mathématiques*, Vol. 43 (1908), p. 179.

†Cf. *Annals of Mathematics*, Vol. 2 (1901), p. 72.

numbers 9, 11, 19 which belong to exponent 2 is $\equiv -1$ (mod 20). This is a special case of the theorem: *The sum of all the incongruent numbers which belong to exponent 2 with respect to any modulus is* $\equiv -1$. This theorem follows directly from the facts that the order of the product of any two operators of order 2 in G is of order 2 and that -1 corresponds to one of these operators of order 2, since it results from these facts that a change of the signs of all the numbers, except -1, which correspond to operators of order 2 in G, does not affect this totality (mod m).

For the same reason it results that the sum of all the numbers, in any reduced system of residues, which belong to an odd exponent or to twice this exponent is always $\equiv 0$. In fact, if we multiply by -1 all those numbers which belong to any odd exponent or to twice this exponent (mod m) we obtain all those which belong to these exponents. From the theorems stated above, it follows directly that all the sum of all the incongruent numbers which belong to exponent δ mod 2^a is $\equiv 0$ or $\equiv -1$ according as $\delta > 2$ or $= 2$. This follows directly from the fact that $\phi(2^a)$ is 2^{a-1}, and hence all the exponents to which odd numbers belong (mod 2^a) are powers of 2.

§ 3. Theorems Relating to a Prime Modulus.

We shall first consider the case when $p-1$, p being the prime modulus, is not divisible by the square of a prime number. Hence we have

$$p-1=p_1p_2 \ldots p_\lambda,$$

where $p_1, p_2 \ldots p_\lambda$ are distinct prime numbers. Since the sum of the distinct roots of the congruence

$$x^{p_a} \equiv 1 \pmod p \qquad a=1, 2, \ldots \lambda$$

is zero, and unity is one of these roots, it results that the sum of the numbers of the series

$$1, 2, \ldots p-1$$

which belong to exponent p_a is $\equiv -1$ (mod p).

To obtain the sum of those numbers which belong to exponent $p_a p_\beta$ ($a, \beta=1, 2, \ldots$ or λ and $a \neq \beta$), we observe that the sum of the roots of the congruence

$$x^{p_a p_\beta} \equiv 1 \pmod p$$

is zero and that the sum of these roots which belong either to exponent p_α or to exponent p_β is -1. Hence the sum of the roots which belong to exponent $p_\alpha\, p_\beta$ is 1. These illustrations suffice to suggest the theorem that the sum of the numbers of the set 1, 2, . . . $p-1$ which belong to an exponent which is the product of an even number of distinct primes is $\equiv 1 \pmod p$, while the sum of these numbers is $\equiv -1 \pmod p$ when the exponent is the product of an odd number of distinct prime factors. We proceed to prove, by complete induction, that this theorem is universally true.

Suppose that this theorem is true for r distinct prime factors p_1, p_2, . . . p_r $(r<\lambda)$, and consider the congruence

$$x^{p_1 p_2 \cdots p_{r+1}} \equiv 1 \pmod p.$$

The sums of the roots which belong to a prime exponent and to an exponent which is the product of two, three . . . up to r distinct primes, are, by hypothesis, as follows:

$$-(r+1) + \frac{(r+1)\,r}{2} - \frac{(r+1)r\,(r-1)}{3!} + \ . \ . \ . \ + (-1)^{r+1}(r+1).$$

This formula results directly from the fact that a cyclic group has one and only one subgroup whose order is an arbitrary divisor of the order of the group and that the $p_1 p_2$ . . . p_{r+1} roots of the congruence

$$p^{p_1 p_2 \cdots p_{r+1}} \equiv 1 \pmod p$$

form a cyclic group $\pmod p$, when they are combined by multiplication.

The terms of the given formula are evidently all the terms, except the first and the last, of the expansion

$$(1-1)^{r+1}.$$

Since the root unity furnishes the first term of this expansion and since the sum of all the roots of the congruence under consideration is $\equiv 0 \pmod p$, it results that the sum of all those roots which belong to exponent $p_1 p_2$ . . . p_{r+1} is congruent to the last term of this expansion. That is, this sum is $\equiv 1 \pmod p$ when $r+1$ is even, while it is $\equiv -1 \pmod p$ when $r+1$ is odd. As the given theorem is true when $r=1$ or 2 it must therefore be universally true.

Suppose now that $p-1$ is divisible by p_α^β, p_α being a prime number and $\beta>1$. There will then be one and only one subgroup of order p_α^β in G and the numbers which correspond to the operators of this subgroup are the roots of the congruence

$$x^{p_\alpha^\beta} \equiv 1 \pmod{p}.$$

Since the sums of these roots which satisfy each of the congruences

$$x^{p_\alpha} \equiv 1 \pmod{p}, \quad x^{p_\alpha^2} \equiv 1 \pmod{p}$$

are zero it results directly that the sum of those roots which belong to exponent p_α^2 is zero. If $\beta>2$, it may be proved in a similar that the sum of those roots which belong to exponent p_α^3 is zero, etc. That is, *the sum of those numbers of the series 1, 2 . . . p—1 which belong to an exponent which is a power, greater than the first, of any prime is* $\equiv 0 \pmod{p}$.

If $p-1$ is divisible by $p_\alpha^\beta q$, q being a prime number, and $\beta>1$, the sum of those roots of the congruence

$$x^{p_\alpha^\beta q} \equiv 1 \pmod{p},$$

which belong to exponent $p_\alpha^2 q$ is $\equiv 0 \pmod{p}$, since the sum of those whose exponents divide $p_\alpha q$ is $\equiv 0 \pmod{p}$. From this it results directly that the sum of those roots which belong to exponent $p^\delta q$, $\delta \overline{\overline{<}} \beta$, is also $\equiv 0 \pmod{p}$, as may be seen by assigning to δ successively the values 3, 4 . . . β. Hence it results from the theorem proved above, by complete induction, that if $p-1$ is divisible by $p_\alpha^\beta q_1, q_2 \ldots q_r$, where $q_1, q_2 \ldots q_r$ are distinct prime numbers, *the sum of the numbers of the series 1, 2 . . . p − 1 which belong to exponent* $p_\alpha^\beta q_1 q_2 \ldots q_r$ *is* $\equiv 0 \pmod{p}$.

We are now in position to prove, by complete induction, that the sum of those numbers of the series 1, 2 . . . $p-1$ which belongs to an exponent which is divisible by the square of a prime number is always $\equiv 0 \pmod{p}$. In fact we may first prove this for the case when this exponent is of the form $q_1^{\alpha_1} q_2^{\alpha_2}$, q_1 and q_2 being distinct primes, and α_1, $\alpha_2>1$. Then we can establish the given result for the case when this exponent is of the form $q_1^{\alpha_1} q_2^{\alpha_2} q_3 q_4 \ldots q_r$ by following the method employed above. After this we establish this result for the exponents which are of the form $q_1^{\alpha_1} q_2^{\alpha_2} q_3^{\alpha_3}$ (α_1, α_2, $\alpha_3>1$), and then for the exponents of the form $q_1^{\alpha_1} q_2^{\alpha_2} q_3^{\alpha_3} q_4 \ldots q_r$. As this process can evidently be continued indefinitely, we have established the theorem: *The sum of those numbers of the series 1, 2 . . . p − 1 which belong to any exponent which is divisible by the square of a prime number is always* $\equiv 0 \pmod{p}$, *while the sum of those which belong to an exponent which is not divisible by the square of a prime is* $\equiv 1$ *or* $-1 \pmod{p}$ *as the number of the prime numbers which divide this exponent is even or odd.**

*A little more general theorem is given by Bachmann, *Niedere Zahlentheorie* (1902), p. 402.

§ 4. A Few Deductions.

The given theorems may be used to advantage to find the numbers which belong to certain exponents, especially when the ϕ-function of these exponents is small. Since the reciprocal of any number belongs to the same exponent as the number itself, it results that when the ϕ-function of this exponent is 2 the number and its reciprocal constitute the only numbers which belong to this exponent. In particular, when the number n belongs to exponent 3(mod p) n^2 must belong to the same exponent and $n^2+n\equiv-1$ (mod p). Hence we may say that a necessary and sufficient condition that the number $n>1$ belongs to exponent 3 (mod p) is that $n(n+1)\equiv-1$. Hence n belongs to exponent 3 whenever p is of the form $n(n+1)+1$. The five primes below 100 which are of this form are 7, 13, 31, 43 and 73. Numbers belonging to exponent 3 (mod p) may often be readily obtained by the following method, whose correctness is easily proved: Find the reciprocal r of 4 (mod p) and find by trial the smallest value of k such that $kp+r-1$ is a perfect square. The two numbers $\frac{1}{2}(p-1)\pm\sqrt{(kp+r-1)}$ will then belong to exponent 3. It is clear that p must have the form $6n+1$.

In a similar manner we see that if n belongs to exponent 4, n^3 will belong to this exponent and $n^3+n\equiv 0(\text{mod } p)$. Two neccessary and sufficient conditions that the number n belongs to exponent 4 (mod p) are therefore that $n(n^2+1)\equiv 0$, and that n is prime to p. Similarly, we observe that a necessary and sufficient condition that n belongs to exponent 6 (mod p) is that $n+\frac{1}{n}\equiv 1$ (mod p). We may deduce from these results the following useful theorem: *A necessary and sufficient condition that the number n, which is prime to the prime odd number p, belongs to exponents 3, 4, or 6 (mod p) is that the sum of n and its reciprocal is congruent to —1, 0, or 1 respectively.* When this sum is congruent to ±1, n must be prime to p. Hence we have that a necessary and sufficient condition that n belongs to exponent 3 or 6 (mod p) is that $n+\frac{1}{n}\equiv-1$ or $\equiv 1$, respectively.

ON THE REPRESENTATION OF AN INTEGER AS THE SUM OF CONSECUTIVE INTEGERS.

By THOMAS E. MASON, Indiana University.

Lucas has shown that every number not of the form 2^n can be expressed as the sum of two or more consecutive positive integers. In this paper we shall consider series of consecutive integers and shall not exclude zero and negative terms. It is proposed to find the number of ways in

which a number may be expressed as a sum of consecutive integers, including the case of a single term.

The question resolves itself into two problems.

Case 1. *When the number of terms in the series is odd.*

In this case we have

$$(A) \qquad m=(2n+1)a,$$

where m is the number, $2n+1$ the number of terms in the series, and a the mid-term. For every such factorization of m there exists a series with m as the sum, and for every series there exists the corresponding factorization. Hence the problem reduces to the problem of finding the number of ways in which m can be factored into two factors, one of which is odd. When the number is 2^n there is only the one odd factor 1. In every other case if we express m thus,

$$m=2^{a} \,.\, p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}.p_4^{\alpha_4}.p_5^{\alpha_5} \;.\;.\;.\; p_r^{\alpha_r},$$

where the p's are distinct odd primes and the α's are the powers to which they occur, then by the theorem for the number of factors of an integer, we have

$$(\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \;.\;.\;.\; (\alpha_r+1)$$

as the number of ways in which we can factor m into two factors, one odd.

Case 2. *When the number of terms in the series is even.*

For this case we have

$$(B) \qquad m=2n(2a+1)/2=n(2a+1),$$

where $2n$ is the number of terms and where a and $a+1$ are the pair of terms in the middle of the series. We see that the problem reduces again to that of finding the number of ways in which m can be separated into two factors, one being odd; the result will evidently be the same as in the first case.

Therefore the total number of ways in which m can be expressed as the sum of consecutive integers is

$$2(\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \;.\;.\;.\; (a_r+1),$$

except in the case where m is 2^n, in which the total number is 2.

Since the factor 2^a has no part in determining the number of ways in which the number may be factored into two factors, one odd, we shall have an infinity of numbers in geometrical series, with ratio two, that can be expressed as the sum of consecutive integers in the same number of ways.

Every factorization of m into two factors, one odd, gives two series. For if $m=(2n+1)a$, we may consider $2n+1$ the number of terms in the series and a the mid-term, or we may take $2a$ as the number of terms and n and $n+1$ as the pair of terms in the middle of the series. Let us consider

$$m=(2n+1)a,$$

where $2n+1$ is the number of terms and a the mid-term. There can be but $a-1$ terms less than a which do not include zero or zero and negative terms. Obviously there will be n terms of the series less than a. Therefore the series will contain all positive terms only when

$$(1) \qquad a-n>0.$$

Let us now consider $2a$ the number of terms of the series and n and $n+1$ the pair of terms in the middle of the series. There can be but $n-1$ terms less than n which do not include zero or zero and negatives. Also there can be but $a-1$ terms of the series below n. Therefore the series will contain only positive terms when

$$(2) \qquad n-a\geqq 0.$$

The relations (1) and (2) are such as to show that when one series contains all positive terms the other will contain a zero or zero and negatives. Since each factorization of m gives one series with all terms positive and one series containing a zero or zero and negatives, we shall have half the total number of series with terms all positive and half with zero or zero and negatives.

From (1) we have that the condition for all positive integers in the series is

$$\text{From (A):} \quad a>n, \quad \frac{m}{2n+1}=a>n, \quad \frac{m}{2n}>\frac{m}{2n+1}>n.$$

$$\therefore \frac{m}{2}>n^2, \text{ or } n<\sqrt{\frac{m}{2}}, \text{ or } 2n<\sqrt{(2m)}.$$

That is, when one less than the number of terms is less than the

square root of twice the number (m) the series consists of positive integers. From (B), in like manner we get

$$2a < \sqrt{(2m)}$$

as the condition for all positive terms, where $2a$ is the number of terms in the series.

Since m can be expressed as the sum of n numbers, it can also be expressed as the sum of n series of consecutive integers. The number of such sets of series will equal the combinations of n things, one being taken from each of n groups, where the number in the group is found by the method of this paper. As m can also be expressed as a difference, a product, or a quotient, it can be expressed as the difference, the product, or the quotient, of series or sets of series.

We shall gather our results into the following

THEOREM. *If we consider a number itself as a series of consecutive integers of one term, and do not exclude zero and negative terms, then a number*

$$m = 2^a \,.\, p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}.p_4^{\alpha_4} \ldots p_r^{\alpha r},$$

where the p's are distinct odd primes and the α's the exponents of the powers to which they appear, can be expressed as the sum of consecutive integers in

$$2(\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \ldots (\alpha_r+1)$$

ways, except in the case $m=2^n$ where the number of ways is 2.

One half the number of series will contain an even number of terms and one half an odd number of terms.

Also one half the number of series will be composed of all positive terms and one half will contain zero or zero and negatives.

An example illustrating the above theorem is the following:

$$15 = 3 \times 5$$

Therefore the number of series will be

$$2(1+1)(1+1) = 2^3 = 8.$$

Number of Terms.	Mid-Term or Pair of Mid-Terms.	The Series.
1	15	15
3	5	4+5+6
5	3	1+2+3+4+5
15	1	−6−5−4−3−2−1+0+1+2+3+4+5+6+7+8
2	7,8	7+8
6	2,3	0+1+2+3+4+5
10	1,2	−3−2−1+0+1+2+3+4+5+6
30	0,1	−14−13− . . . +0+1 . . . +14+15

NOTE ON PRIME NUMBERS.

By DERRICK N. LEHMER, University of California.

It is a well known theorem that it is possible to find an arbitrarily great number of consecutive composite numbers. This appears from the values which the expression $n!+r$ takes for $r=2, 3, \ldots, n$. This theorem furnishes an interesting proof of the theorem that the number of primes less than or equal to x is not determined by a function of x which is a polynomial in x of finite degree. For if $f(x)$ were such a function of degree n, then for $x=(n+2)!+r$, $f(x)$ must keep the same value for $r=2, 3, 4, \ldots, n+2$. If this value is k, then $f(x)-k=0$ is an equation of degree n with $n+1$ roots, which is impossible.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

362. Proposed by JAMES F. LAWRENCE, Stillwater, Okla.

Show that the number of solutions in positive integers, zero included, of the equation $x+2y+3z=6n$, is $3n^2+3n+1$.

Solution by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Burghfield, England.

$x+2y+3z=6n$. z may have any value from 0 to $2n$, inclusive.

Hence we may assign to it any even or odd value from 0 to $2n$, inclusive.

i. Let $z=2r$. $x+2y=6(n-r)$. y may have any value from 0 to $3(n-r)$.

$\therefore$ There are $3(n-r)+1$ solutions when z is $2r$.

$\therefore$ Total number of solutions for z even is

$$\sum_{r=0}^{r=n}[3(n-r)+1]=(3n+1)(n+1)-\frac{3n.(n+1)}{2}$$

$$=\frac{n+1}{2}(6n+2-3n)=\frac{(n+1)(3n+2)}{2}.$$

ii. Let $z=2r+1$. $x+2y=6(n-r)-3$. y may have any value from 0 to $3(n-r)-2$.

$\therefore 3(n-r)-2+1=3(n-r)-1$ solutions.

$\therefore$ Total number of solutions when z is odd:

$$=\sum_{r=0}^{r=n-1}[3(n-r)-1]=(3n-1)n-\frac{3n(n-1)}{2}=\frac{n}{2}(6n-2-3n+3)=\frac{n}{2}(3n+1).$$

$\therefore$ The total number of solutions

$$=\frac{(n+1)(3n+2)}{2}+\frac{n(3n+1)}{2}=\frac{3n^2+5n+2+3n^2+n}{2}=3n^2+3n+1.$$

Also solved by H. Prime, J. Scheffer, H. C. Feemster, and A. M. Harding.

363. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

(a) If a and n be positive integers, the integral part of $[a+\sqrt{(a^2-1)}]^n$ is odd.

(b) If a and n be positive integers, the integral part of $[\sqrt{(a^2+1)}+a]^n$ is odd when n is even and even when n is odd. [From Todhunter's *Algebra*, p. 353].

I. Solution by the PROPOSER.

Proof. (a) Let $[a+\sqrt{(a^2-1)}]^n=P+Q\sqrt{(a^2-1)}=m$.

Then $[a-\sqrt{(a^2-1)}]^n=P-Q\sqrt{(a^2-1)}=1/[a+\sqrt{(a^2-1)}]^n$.

$\therefore 0<P-Q\sqrt{(a^2-1)}<1$.

Adding m to each member of the inequality $m<2P<m+1$.

Therefore, the integral part of m is odd.

(b) Let $[\sqrt{(a^2+1)}+a]^n=R+S\sqrt{(a^2+1)}=k$.

Then $[-\sqrt{(a^2+1)}+a]^n=R-S\sqrt{(a^2+1)}=\left(\frac{-1}{\sqrt{(a^2+1)}+a}\right)^n$.

If n is even, $0<R-S\sqrt{(a^2+1)}<1$. Adding k, $k<2R<k+1$. Whence the integral part of k is odd.

If n is odd, $-1<R-S\sqrt{(a^2+1)}<0$. Adding k, $k-1<2R<k$. Whence the integral part of k is even.

Also solved similarly by S. Lefschetz.

II. Solution by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Burghfield, England, and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let I be the integral part of $[a+\sqrt{(a^2-1)}]^n$.

Then $I+F=[a+\sqrt{(a^2-1)}]^n$, where F is a proper fraction,

$$=a^n+na^{n-1}\sqrt{(a^2-1)}+\frac{n.(n-1)}{2!}(a^2-1)+\frac{n.(n-1)(n-2)}{3!}(a^2-1)^{\frac{3}{2}}+\ldots$$

Also, $a>\sqrt{(a^2-1)}$; $\therefore a-\sqrt{(a^2-1)}<1$, and $[a-\sqrt{(a^2-1)}]^n<1$.

Let $F'=[a-\sqrt{(a^2-1)}]^n=a^n-na^{n-1}\sqrt{(a^2-1)}$

$$+\frac{n.(n-1)}{2!}(a^2-1)-\frac{n.(n-1)(n-2)}{3!}(a^2-1)^{\frac{3}{2}}+\ldots$$

$$\therefore I+F+F'=2\left(a^n+\frac{n.(n-1)}{2!}(a^2-1)+\ldots\right)=2p \text{ (say)}.$$

Hence, $F+F'=1$, and $I=2p-1$ and is odd.

Similarly, $I+F=[\sqrt{(a^2+1)}+a]^n=(a^2+1)^{\frac{1}{2}n}+n(a^2+1)^{\frac{1}{2}(n-1)}$

$$+\frac{n.(n-1)}{2!}(a^2+1)^{\frac{1}{2}(n-1)}a^2+\ldots$$

$$F'=[\sqrt{(a^2+1)}-a]^n=(a^2+1)^{\frac{1}{2}n}-n(a^2+1)^{\frac{1}{2}(n-1)}a+\ldots$$

(i) n odd. $I+F-F'=2[n(a^2+1)^{\frac{1}{2}(n-1)}a+\ldots]=2p$ (say) $=$ an even integer.

$\therefore F-F'=0$, and I is an even integer.

(ii) n even. $I+F+F'=2\left[(a^2+1)^{\frac{1}{2}n}+\frac{n.(n-1)}{2!}(a^2+1)^{\frac{1}{2}(n-2)}a^2+\ldots\right]=$ an even integer.

$\therefore F+F'=1$, and I is an odd integer.

GEOMETRY.

389. Proposed by H. PRIME, Boston, Mass.

On the same side of a given base, triangles are erected such that the bisectors of their vertex angles all pass through a given point. Find the locus of the vertices (i) when the vertex angle are all equal, (ii) when the vertex angles are all unequal.

Solution by the PROPOSER.

Take the given base $BB=2m$ for the axis of x, its midpoint O for the origin. Let the triangles be erected on the positive side of BB; a, b be the given point, P any point of the required locus, and the circle BPB, radius r, cut OY' in Q. The coördinates of Q are 0, $\pm\sqrt{(r^2-m^2)}-r$. The angle bisector passes through Q and a, b. Its equation is

$$y=[b\mp\sqrt{(r^2-m^2)}+r]x/a\pm\sqrt{(r^2-m^2)}-r\ldots(1).$$

The equation of the circle BQB is

$$x^2+[y\mp\sqrt{(r^2-m^2)}]=r^2\ldots(2).$$

The upper sign apply when the center of the circle is above the axis of x, the lower sign when it is below.

Eliminating the variable parameter r between (1) and (2), we have

$$\text{(A)}\quad 2ay^2+2bxy+(a-x)(x^2+y^2-m^2\mp\sqrt{[(x^2+y^2-m^2)^2+4m^2y^2]},$$
$$\text{or, } y(ay-bx)^2-(a-x)(x^2+y^2-m^2)(ay-bx)=m^2y(a-x)^2.$$

This is the general equation of the locus and defines its form in every relation of the constants. If $a=0$, (A) reduces to either

$$\text{(B)}\qquad x=0, \text{ or,} \qquad \text{(C)}\qquad x^2+y^2-y(b^2-m^2)/b=m^2.$$

(B) applies when the triangles are isosceles, their vertex angles unequal and bisected by the axis of y.

(C) is the equation of a circle whose radius$=(b^2+m^2)/2b=$a constant. Hence all the triangles are inscribed in the same circle and, having the same base, their vertex angles are equal.

Also solved by C. N. Schmall.

390. Proposed by PROF. R. C. ARCHIBALD, Brown University, Providence, R. I.

Find, geometrically and without introducing focal properties, the locus of the vertices of the conjugate parallelograms of an ellipse.

Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Let $y=mx+\sqrt{(a^2m^2+b^2)}$, $y=m'x+\sqrt{(a^2m'^2+b^2)}$ be the equation of the tangents parallel to two conjugate diameters. But between m and m' there is the well known relation: $mm'=-(b^2/a^2)$, whence $m'=-(b^2/a^2m)$. Substituting this in the second of the above equations, we have $a^2ym+b^2x=ab\sqrt{(a^2m^2+b^2)}$, and dividing the last equation by $y-mx=ab\sqrt{(a^2m^2+b^2)}$, we obtain

$$m=\frac{b}{a}\cdot\frac{ay-bx}{ax+bx};$$

and substituting in the latter equation, we finally get $a^2y^2+b^2x^2=2a^2b^2$, the equation of a concentric ellipse with the semi-axes $a\sqrt{2}$ and $b\sqrt{2}$.

391. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Burghfield, England.

An ellipse is inscribed in the triangle of reference and has one focus at ($\sec A$, $\sec B$, $\sec C$). Find the other focus and the sum of the squares of the axes of the ellipse.

Solution by WILLIAM HOOVER, Ph. D., Athens, Ohio.

In trilinear coördinates, let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ be the two foci. Then the first is found by

$$\alpha_1\cos A=\beta_1\cos B=\gamma_1\cos C\ldots(1),$$
$$\text{and } a\alpha_1+b\beta_1+c\gamma_1=2\triangle\ldots(2),$$

$$\text{or, } \alpha_1=2R\cos B\cos C,\ \beta_1=2R\cos A\cos C, \text{ and } \gamma_1=2R\cos A\cos B\ldots(3),$$

R being the radius of the circum-circle.

If b_1 be the semi-minor axis of the ellipse,

$$\alpha_1\alpha_2=b_1^2=\beta_1\beta_2=\gamma_1\gamma_2\ldots(4), \text{ or,}$$
$$\cos B\cos C.\alpha_2=\cos A\cos C.\beta_2=\cos A\cos B.\gamma_2\ldots(5),$$

which with (3) gives

$$\alpha_2=R\cos A,\ \beta_2=R\cos B,\ \gamma_2=R\cos C\ldots(6).$$

(4) now gives $b_1^2=2R^2\cos A\cos B\cos C\ldots(7)$.

Also, d=the distance between the ortho-center and circum-center, is given by

$$d^2=R^2(1-8\cos A\cos B\cos C)\ldots(8).$$

If $2a_1$ be the major axis of the ellipse, $a_1^2=\frac{1}{4}d^2+b_1^2=\frac{1}{4}R^2$, or $2a_1=R$, and determining $4a_1^2+4b_1^2$.

392. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

A tangent to a curve at any point P cuts the tangent and the normal at a fixed point O in the points M and N, and the rectangle $OMP'N$ is completed. Find the curve which is such that the triangle formed by the tangents at any three points P, Q, R is equal to the triangle formed by the corresponding points P', Q', R'.

No solution of this problem has been received.

CALCULUS.

316. Proposed by C. N. SCHMALL, New York City.

$$\int_0^\infty \frac{\cos ax}{1+x^2}dx=\tfrac{1}{2}\pi e^{-a}=\int_0^\infty \frac{x\sin ax}{1+x^2}dx.$$

(From Bromwich, *Theory of Infinite Series*, p. 442, ex. 5, and also from Carslaw, *Fourier's Series*, p. 113, ex. 12.) Prove this by any method.

II. Solution by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Burghfield, England.

Lemma. $u=\underset{h\doteq 0}{L}[\sin h+\frac{1}{2}\sin 2h+\frac{1}{3}\sin 3h+\ldots]=\underset{h\doteq 0}{L}\frac{\pi-h}{2}$, the sum being taken between 2π and small values of h, $=\frac{1}{2}\pi$.

$\therefore u=\int_0^\infty \frac{\sin x}{x}dx=\frac{1}{2}\pi$, and it is also clear that $\int_0^\infty \frac{\sin ax}{x}dx=\frac{1}{2}\pi$.

Now let $U=\int_0^\infty \frac{x\sin ax\, dx}{1+x^2}$; $U-\frac{1}{2}\pi=\int_0^\infty \frac{x\sin ax\,dx}{1+x^2}-\int_0^\infty \frac{\sin ax}{x}$

$$=-\int_0^\infty \frac{\sin ax}{x(1+x^2)}dx\ldots(1).$$

Differentiating twice with respect to a,

$$\frac{d^2U}{da^2}=\int_0^\infty \frac{x\sin ax}{1+x^2}dx=U.$$

Multiplying by $\frac{dU}{da}$, $\frac{d^2U}{da^2}\cdot\frac{dU}{da}=\frac{U.dU}{da}$. $\frac{1}{2}\frac{d}{da}\left(\frac{dU^2}{da^2}\right)=\frac{1}{2}\frac{d(U^2)}{da}$, $\left(\frac{dU}{da}\right)^2=U^2+\kappa$,

$$\frac{dU}{\sqrt{(U_2+\kappa)}}=da.$$

$\therefore \log[U+\sqrt{(U^2+\kappa)}]=a+\lambda,\ U+\sqrt{(U^2+\kappa)}=e^{a+\lambda}$.

Also, $\sqrt{(U^2+\kappa)}-U=\kappa e^{-(a+\lambda)}$, $2U=e^{a+\lambda}-\kappa e^{-(a+\lambda)}=Ce^{-a}+C'e^{a}$, where C, C', are constants. U not increasing indefinitely with a it follows that $C'=0$. When a is very small, (1) becomes

$$\underset{a\doteq 0}{L}\, U-\tfrac{1}{2}\pi=\underset{a\doteq 0}{L}\, -\int_0^\infty \frac{adx}{1+x^2}=\underset{a\doteq 0}{L}\, -\frac{a\pi}{2}=0;\ \therefore C=\tfrac{1}{2}\pi, \text{ and}$$

$$u=-\int_0^\infty \frac{x\sin axdx}{1+x^2}=\frac{\pi}{2}e^{-a} \ (a \text{ being positive}).$$

But $\displaystyle\int_0^\infty \frac{x\sin ax}{1+x^2}dx=\frac{\pi}{2}-u=\frac{\pi}{2}(1-e^{-a})$.

Differentiating with respect to a,

$$\int_0^\infty \frac{x\cos ax}{1+x^2}dx=\frac{\pi}{2}e^{-a}.$$

$\therefore$ etc. (Cf. Roberts' *Treatise on the Integral Calculus*, Part I, p. 181.)

317. Proposed by C. N. SCHMALL, New York City.

A generating line of a right circular cylinder passes through the center of a sphere. The diameter of the cylinder is less than the radius of the sphere. Show that the surface of the cylinder included within the sphere is given by an elliptic integral.

Solution by A. M. HARDING, Fayetteville, Arkansas.

Let a=diameter of cylinder; r=radius of sphere. Choose the generating line of cylinder for z-axis. Let equation of sphere and cylinder be

$$x^2+y^2+z^2=r^2 \text{ and } x^2+y^2=ax,$$

respectively. Then

$$\frac{A}{4}=\int\int\left[1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2\right]^{\frac{1}{2}}dzdx.$$

Eliminate y and obtain $z^2+ax=r^2$. Hence z-limits are 0 and $\sqrt{(r^2-ax)}$, x-limits are 0 and a.

From equation of cylinder, we find

$$\frac{\partial y}{\partial x}=\frac{a-2x}{2y},\quad \frac{\partial y}{\partial z}=0.$$

$$\therefore \frac{A}{4}=\int_0^a \int_0^{\sqrt{(r^2-ax)}} \left[1+\left(\frac{a-2x}{2y}\right)^2\right]^{\frac{1}{2}} dzdx=\int_0^a \int_0^{\sqrt{(r^2-ax)}} \frac{a}{2\sqrt{(ax-x^2)}} dzdx,$$

since $y^2=ax-x^2$.

$$\therefore A=2a\int_0^a \sqrt{\left(\frac{r^2-ax}{ax-x^2}\right)}dx. \quad \text{Putting } x=a\sin^2\phi,$$

$$A=2a\int_0^{\frac{1}{2}\pi} \frac{\sqrt{(r^2-a^2\sin^2\phi)}}{\sqrt{(a^3\sin^2\phi-a^2\sin^4\phi)}}.2a\sin\phi\cos\phi\, d\phi$$

$$=4a\int_0^{\frac{1}{2}\pi} \sqrt{(r^2-a^2\sin^2\phi)}.\frac{a\sin\phi\cos\phi}{a\sin\phi\sqrt{(1-\sin^2\phi)}}d\phi$$

$$=4ar\int_0^{\frac{1}{2}\pi} \sqrt{\left(1-\frac{a^2}{r^2}\sin^2\phi\right)}d\phi=4ar\,E\left(\frac{a}{r},\frac{\pi}{2}\right),\ a<r.$$

Also solved by Francis Rust and J. Scheffer.

318. Proposed by JOHN C. GREGG, Greencastle, Ind.

A thread is wound spirally n times around a cone, the radius of whose base is r, and slant height h, the turns being at uniform distance apart. If the thread is kept taut, what will be the length of the trace of its end on a horizontal plane?

No correct solution of this problem has been received. Professor Feemster has given a solution finding the length of the thread. But the problem does not require that. Let us have a number of solutions of this problem. Ed. F.

319. Proposed by C. N. SCHMALL, New York City.

Given $u=\frac{yz}{x}$, $v=\frac{zx}{y}$, $w=\frac{xy}{z}$, prove

$$\begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y}, & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y}, & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x}, & \frac{\partial w}{\partial y}, & \frac{\partial w}{\partial z} \end{vmatrix}=4xyz.$$

Solution by S. LEFSCHETZ, Ph. D., The University of Nebraska.

If $\triangle$ is the determinant proposed, we have:

$$\triangle=\begin{vmatrix} -\frac{yz}{x^2}, & \frac{z}{x}, & \frac{y}{x} \\ \frac{z}{y}, & -\frac{xz}{y^2}, & \frac{x}{y} \\ \frac{y}{z}, & \frac{x}{z}, & -\frac{xy}{z^2} \end{vmatrix}=xyz\begin{vmatrix} -\frac{1}{x}, & \frac{1}{y}, & \frac{1}{z} \\ \frac{1}{x}, & -\frac{1}{y}, & \frac{1}{z} \\ \frac{1}{x}, & \frac{1}{y}, & -\frac{1}{z} \end{vmatrix}=xyz\begin{vmatrix} 0, & 0, & \frac{2}{z} \\ \frac{2}{x}, & 0, & 0 \\ \frac{1}{x}, & \frac{1}{y}, & -\frac{1}{z} \end{vmatrix}=4.$$

The value announced, $4xyz$, is therefore erroneous.

Solved similarly by J. Scheffer and W. J. Greenstreet.

MECHANICS.

356. Proposed by the late G. B. M. ZERR, Ph. D.

A cantilever beam length a is loaded with c pounds per running foot at its fixed end and increases uniformly to b pounds per running foot at its free end. Find the deflection at the free end due to this load.

Solution by FRANCIS RUST, E. E., Pittsburg, Pa.

The problem is solved by determining the elastic curve of the beam's axis from its second differential equation

$$y''=\frac{M}{IE},$$

as derived from Hooke's law, to be found in all text books. M is the bending moment in the point, abscissa$=x$; I is the moment of inertia of the beam's cross-section, and E is the modulus of elasticity of its material.

M is to be determined by the arrangement of the load with mathematical certainty. Taking the point, abscissa$=x$, for origin,

$$M=\int_0^{a-x} q\xi\, d\xi,$$

q, the load per lineal foot, to be expressed as a function of the variable of integration ξ. $q=c+px$, and $c+pa=b$. Consequently,

$$p=\frac{b-c}{a},$$

and q as a function of ξ is

PROBLEMS FOR SOLUTION.

ALGEBRA.

369. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

If $f(m)=(1+x)^m$, and $f(n)=(1+x)^n$, why not obviously $f(m).f(n)=(1+x)^{m+n}=f(m+n)$?

370. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Prove that, if the fraction m/p (p prime) gives a recurring decimal with an even number of digits in the cycle, the sum of the two halves will be composed of 9's. (The special case where the number$=p-1$ was proposed in the MONTHLY, Vol. IV, and answered in Vol. V, p. 11. The proof there given, however, is not complete.) The above property is true of other fractions, *e. g.*, $\frac{1}{77}=.\dot{0}1298\dot{7}$, $\frac{1}{133}=\frac{1}{7.19}=.\dot{0}07518796,99248120\dot{3}$. Find for what fractions this is true.

371. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

In a G. P. of an odd number of terms, all of the terms being positive, and the ratio different from 1, show that the middle term is less than the arithmetical mean.

372. Proposed by S. LEFSCHETZ, Ph. D., University of Nebraska.

Prove that $\sum_{n=1}^{n=\infty} n^2x^{n-1}=\frac{1+x}{(1-x)^3}$, if mod.$x<1$. (*Schlömilch.*)

373. Proposed by X, National Electric Light Association, Brooklyn, N. Y.

(a) For underground distribution of direct current electrical energy, we have $DA^n-CB^n=H$, where the only unknown is n, which represents the number of years it will take a large direct current low tension feeder to pay by line loss saving for the increased investment over a smaller feeder.

(b) $bVf^2l^2\beta^2+aVf\beta^{1.6}=W$, the iron loss equation which is to be solved for β. When $bVf^2l^2\beta^2$ represents the eddy current loss in the core of a transformer, and $aVf\beta^{1.6}$is the hysteresis loss in the core. a, b, and l are constants of the core, V is the voltage, f is the frequency, and β is the flux density and is the only unknown in the equation.

Letting $bVf^2l^2=A$, and $aVf=C$, we have $A\beta^2+C\beta^{1.6}=W$.

GEOMETRY.

401. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Find by Euclidean geometry a point whose distances from the vertices of an equilateral triangle are in the ratio 3:4:5. The general case of ratio $a:b:c$ would prove interesting.

402. Proposed by H. PRIME, Boston, Mass.

The diameter of a hoop-shaped ring (or collar) is 24 inches at one edge and 28 inches at the other edge. A cross-section is a crescent with circular arcs of 120° and 60°, whose common chord is 4 inches long. Find its volume by elementary methods (without the use of calculus or the center of gravity).

CALCULUS.

322. Proposed by E. B. ESCOTT, University of Michigan.

Find the equation of the curve such that the solid of revolution generated by revolving it about the x-axis shall have a volume equal to the m/nth part of the volume of the circumscribed cylinder.

323. Proposed by C. N. SCHMALL, New York City.

From what height must an elastic ball be dropped in order that, after impact with the hard surface of the sidewalk, it may rebound to a given altitude a in the least possible time from the moment of descent?

324. Proposed by V. M. SPUNAR, Chicago, Ill.

A parabola slides between two rectangular axes; find (a) the locus of the focus, and (b) the locus of the vertex.

NOTES AND NEWS.

In the May, 1911, number of the *Mathematical Gazette*, page 93, there occurs the following statements: "If modern American text-books in arithmetic offer any criterion of the teaching of the subject in the States, it would appear that America lags behind this country (England) and, indeed, most European countries. For the most part they are characterized by a drab monotony of treatment, by a lack of vitality, and by an entire absence of any correlation with other branches of a mathematical education." M.

Macmillan and Company recently issued volume II of the *Theory of Determinants* in the historical order of development by Dr. Thomas Muir, Superintendent of Education in Cape Colony, Africa. This volume covers the period from 1841 to 1860 and is very similar to volume I in general arrangement. It shares also with volume I the decided defect that the articles quoted are not always assigned to the year of publication but they frequently bear an earlier date, in case such a date appears at the end of the article. In a historical work this arrangement is very confusing since it is not in accord with the present common practice. For instance, if one reads, on page 81 of this book, that the second edition of the "Elementary theorems relating to the determinants, by Spottiswoode, appeared in Crelle's Journal in 1853, and then turns to page 46, volume I, of the *Encyklopaedie der Mathematischen Wissenschaften* and finds the date 1856 assigned to the same article, one is naturally annoyed. Some of the most careful works of reference, like the *Encyclopédie des Sciences Mathématiques*, give both the date of publication and also the date when the article is supposed to have been completed, in case the latter is known. When only one date is given it should always be the former, according to modern practice. M.

BOOKS.

A History of the Theories of Aether and Electricity From the age of Descartes to the close of the nineteenth century. By E. T. Whittaker, Hon. Sc. D. (Dubl.); F. R. S.; Royal Astronomer of Ireland. 8vo. Cloth, xiii+475 pages. Price, $4.50. New York and London: Longmans, Green, & Co.

The author has given in this book a very complete history of the various theories respecting the luminiferous ether held by all the great thinkers since the time of Descartes. In most instances where mathematical treatment is necessary the author uses Vector Analysis, thus making some parts difficult to read for those who are unfamiliar with that branch of mathematics. An idea of the scope of the work may be obtained from the table of contents. The various subjects come under twelve chapters, as follows: Chapter I, The Theory of the Aether in the 17th Century; Chapter II, Electric and Magnetic Science, prior to the Introduction of Potentials; Chapter III, Galvanism, from Galvani to Ohm; Chapter IV, The Luminiferous Medium, from Bradley to Fresnel; Chapter V, the Aether as an Elastic Solid; Chapter VI, Faraday; Chapter VII, The Mathematical Electricians of the Middle of the Nineteenth Century; Chapter VIII, Maxwell; Chapter IX, Models of the Aether; Chapter X, The Followers of Maxwell; Chapter XI, Conduction in Solutions and Gases, from Faraday to J. J. Thomson; Chapter XII, The Theory of Aether and Electrons in the Closing Years of the 19th Century.

Every teacher of Physics will want this thoroughly scholarly work in his library. F.

Advanced Calculus. By Edwin Bidwell Wilson, Associate Professor of Mathematics, Massachusetts Institute of Technology. 8vo. Cloth, ix+566 pages. Price, $5.00. Boston and Chicago: Ginn & Co.

"Professor Wilson's Advanced Calculus supplies in a single volume a comprehensive second course in calculus. Although modern rigorous tendencies are given due attention, the chief aim of the book is to confirm and to extend the student's knowledge of the great formal methods of analyais that are essential alike to the practical and to the pure mathematician. To connect with elementary texts, two chapters in review are supplied, and many subsequent chapters are tempered with material which is essentially review. Advanced differential calculus is represented by work on Taylor's formula, with special reference to approximate analysis, partial differentiation of explicit and implicit functions, complex numbers, and vectors. As an extension of previous formal work in integration, four chapters are given to the integration of differential equations. In integral calculus line integrals, multiple integrals, infinite integrals, special functions defined by integrals, and the calculus of variations are treated. Then follow chapters on series, special develments, functions of a complex variable, elliptic functions, and functions of real variables. Throughout the work especial attention has been paid to the needs of students of applied mathematics and mathematical physics. A very large number of exercises have been provided, and every attempt has been made to furnish a thorough, practical, teachable, live textbook and laboratory manual of higher calculus."

The publishers of this work have again put teachers of the Calculus under great obligation to them by bringing out this splendid treatise which cannot be of great advantage commercially, but which will be of immense value pedagogically. They, as well as its author, deserve the thanks of American mathematicians. F.

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SOME USEFUL MATHEMATICAL BOOKS BEYOND ELEMENTARY CALCULUS.

By DR. G. A. MILLER, University of Illinois.

There is probably always a considerable number of young men who would like to extend their knowledge of mathematics but who are in doubt as to the most useful books. This applies especially to those who do not have easy access to good libraries and who have never enjoyed good educational advantages. The following suggestions are intended mainly for this class, but it is hoped that some others may be able to derive profit from reading them. The fact that there are so many good recent American mathematical books is especially encouraging, and one of our objects is to call attention to this important fact.

We first inquire, what are the most useful works along the line of algebra for a student who has had only an elementary course in this subject. Doubtless all would agree that such a student should secure Bôcher's *Introduction to Higher Algebra*, published by the Macmillan Company in 1907. He may find Bôcher's treatment of many subjects limited and too concise, and it is hoped that he will be led to look elsewhere for greater details as well as for algebraic subjects which he does not find in this work; but he will probably appreciate the methods and the subject matter of this work more and more as he advances.

In particular, he may desire to learn something about the fundamental properties of numbers. In this case he would probably find Reid's *Theory of Algebraic Numbers*, published by the Macmillan Company in 1910, very inspiring as well as very elementary. Cajori's *Theory of Equations*, published by the same company in 1904, would serve very well to secure a deeper knowledge of the fundamental subject of equations, and this knowledge could be extended along important lines by reading Dickson's *Theory of Algebraic Equations*, published by Wiley and Sons in 1903. Among tne classic algebras in foreign languages the following are especially worth having: Weber's *Lehrbuch der Algebra*, second edition, published at Braunschweig, Germany, by Vieweg und Sohn, 1898-1908, three volumes of 703, 855, and

733 pages, respectively; Serret's *Algèbra Supérieure,* sixth edition, published at Paris, France, by Gauthier-Villars, 1910, two volumes of 648 and 694 pages, respectively; Capelli's *Istituzioni di Analisi Algebrica,* fourth edition, published at Napoli, Italy, by Pellerano, 1909, one volume of 953 pages.

The student who wishes to extend his knowledge of analysis beyond a first course in elementary calculus will find an abundance of good books suitable to various grades of advancement. Wilson's *Advanced Calculus,* recently published by Ginn and Company, and Hedrick's translation of Goursat's *Course in Mathematical Analysis,* Volume I, published by the same company, would doubtless serve to guide the student wisely on entering this vast and interesting field of mathematics. The two little volumes on differential equations by Cohen, published by Heath and Company, can also be heartily recommended in view of the great importance of the field to which they provide an easy introduction.

The vast field of function theory may be wisely entered either by means of volume I of the *Lectures on the Theory of Functions of Real Variables* by Pierpont, published a few years ago by Ginn and Company, or by means of the second edition of Forsyth's *Theory of Functions of a Complex Variable,* published by the Cambridge, England, University Press in 1900. The two books by Harkness and Morley, bearing the titles *Introduction to the Theory of Analytic Functions,* and *Treatise on the Theory of Functions,* published by Macmillan and Company, are also very good books for the beginner in this field.

Among the classic works on analysis in foreign languages, Jordan's *Cours d' Analyse,* published by Gauthier-Villars, Paris, France, and Picard's *Traité d' Analyse,* published by the same firm, are very celebrated, but there is a large number of other works along this line which deserve high praise, and which are extensively used. Osgood's *Lehrbuch der Funktionentheorie,* written by an American but published in German by Teubner of Leipzig, in 1907, constitutes one of the best introductions to this theory, and furnishes the shortest routes to many important questions.

The student who desires wise guidance in pursuing geometry beyond an elementary course in analytic geometry may safely begin with the *Projective Geometry* by Veblen and Young, published by Ginn and Company in 1910. Like Bôcher's *Algebra* this work will probably be appreciated more and more as the student advances, and it would be desirable to seek elsewhere for more complete information on many of the separate subjects. The *Introduction to Projective Geometry* by Emch, published by Wiley and Sons in 1905, could render valuable services along this line. Students who wish to enter the important field of differential geometry could find wise guidance in the *Treatise on Differential Geometry* by Eisenhart, published lately by Ginn and Company; and those who wish to enter the more special but far reaching field of projective differential geometry will naturally begin with Wilczynski's *Projective Differential Geometry,* published by Teubner of Leipzig, Germany, in 1906.

Among the classic works on geometry in foreign languages *La Thèorie Gênerale des Surfaces* by Darboux, published by Gauthier-Villars, Paris, France, is one of the most noted. This consists of four large volumes, and was published from 1837 to 1896. The second edition of Bianchi's *Legioni di Geometria Differenziale*, 1902-03, in two volumes of 523 and 594 pages, respectively, published by Spoerri, Pisa, Italy, is also very well known.

The preceding remarks relate separately to the three great fields of pure mathematics,—algebra, analysis, and geometry. A great deal of the most important mathematical literature does not limit itself to any one of these fields. Moreover, the student of mathematics needs outlook and independence as soon as he has sufficient knowledge to use these. That is, he needs books on books as well as thoughts on thoughts and ideas on ideas. Fortunately, such literature is growing very rapidly, although the bulk of it is in foreign languages. One of the most useful aids along this line in English is the Subject Index, Volume I, Pure Mathematics, of the Royal Society of London *Catalogue of Scientific Papers*, 1800-1900, published by the Cambridge, England, University Press, in 1908. This volume of about 700 pages is a subject index of the mathematical articles which appeared in 700 different serials during the nineteenth century, and is said to contain 38,748 entries.

Although this Index gives only the places where articles relating to different subjects were published it is of great value as one can often learn much in reference to what is probably new and what has been done by others from the various subjects to which articles relate and from the extent of these articles. The student of mathematics should never lose sight of the fact that good books are his tools and that it is almost as necessary for him to have the proper books as for the mechanic to have the proper tools. Only those who employ the best available tools for the work in hand have reason to expect notable success.

The Index mentioned above has been continued since 1900 by the annual mathematical volume in the *International Catalogue of Scientific Literature*, published for the International Council by the Royal Society of London. Each of the seventeen separate annual volumes of this catalogue contains both a subject and an author index, but it does not furnish more information about the article than the title conveys. The only extensive mathematical work which gives critical reviews of articles appearing from year to year is the excellent German publication entitled, *Jahrbuch über die Fortschritte der Mathematik*. About sixty different mathemations coöperate in providing reviews for this publication, which has appeared practically each year since 1871, and constitutes a most valuable source of information. The number of different articles reviewed at present is about three thousand per year and about one-third of a page is devoted, on an average, to each review.

As these reviewers are specialists of high scientific standing they have produced by their combined efforts a monumental work, which has had a most salutary influence on the development of mathematics, as it encour-

ages publications of high order and discourages research publications of little or no scientific value, as the latter publications are exposed to unfavorable reviews in this well known annual. It would be very desirable to have at least one more such annual, since even men of high scientific standing are not always free from prejudice. On the whole, it must be said that the reviews in the *Fortschritte* have been excellent and have more often dealt too leniently than too severely with the articles under consideration.

The most magnificent direct coöperation of mathematicians to produce a great work has been called into existence for the sake of completing the great German and French mathematical encyclopedias which are now in the course of publication by the firms of Teubner of Leipzig, Germany, and Gauthier-Villars of Paris, France. According to a recent circular more than 160 different mathematicians have been working on the German edition and more than 100 on the French. The vastness of the work may be inferred from the fact that the published parts of the German edition would fill more than a score of volumes of four hundred pages each although a large part of the field has not yet been covered; and, judging from the parts of the French that have appeared, this edition will be at least twice as large as the German. For instance, the number of pages in the published parts of Volume I of both of these two editions are as follows; the first number applying to the German edition. Fundamental principles of arithmetic, 27, 62; combinatory analysis and determinants, 19, 70; irrational numbers and convergence of infinite processes with real numbers, 100, 196; ordinary and higher complex numbers, 37, 140; infinite algorithms with complex numbers, 8, 20; theory of sets, 24, 42; finite discrete groups, 19, 85.

The object of this great work of reference is to give as completely as possible the fully established mathematical results and to exhibit by means of careful references the historical development of mathematical methods since the beginning of the nineteenth century. The work is not restricted to the so-called pure mathematics, but it includes applications to mechanics, physics, astronomy, geodesy, and various other technical subjects, so as to exhibit *in toto* the position occupied by mathematics in the present state of our civilization.

While such a vast work will naturally appeal to those interested in higher mathematics more than to those whose main interests are confined to the more elementary subjects, yet the latter will find much that is within their easy comprehension, especially in the introductory parts of arithmetic, geometry, and algebraic analysis. Moreover, it is of considerable value to read things once in a while which are not within one's easy comprehension. Great thoughts can sometimes be enjoyed even by those who cannot comprehend them completely, and a superficial view of the vastness of the developed parts of mathematics is much better than total ignorance and frequently awakens an interest in a particular field which appears to have received relatively too little attention. It may also serve to call attention to the im-

portant problem of mathematical transportation, or the utilization of the developments of one field in other fields, and the bringing together of facts whose similarity becomes striking through proximity. All the better libraries should be induced to subscribe for at least one of these great encyclopedias since such great works are of permanent value. Descriptive circulars can be obtained gratis by addressing the publishers as well as from some importers.

Among the less extensive works giving an outline of the main fields of mathematics which have been developed, the second edition of Pascal's *Repertorium der hoeheren Mathematik* takes the foremost place. The first two volumes, each covering more thon 500 pages, and costing about two dollars and a half, appeared in 1910, and the remaining two volumes are expected to appear towards the end of the present year. This work is also published by B. G. Teubner of Leipzig, Germany. In fact, this firm is now publishing more works on advanced mathematics than any other firm in the world.

Another very useful general work, which is, however, less modern in spirit, is Hagen's *Synopsis der hoeheren Mathematik*, consisting of three quarto volumes of about 400 pages each. This work was prepared in America when Hagen was Director of the Observatory of Georgetown College, but it was published in Germany by Dames, of Berlin. It may be of interest to observe that while the number of people who use the English language is very much larger than the number of those who use German, a few American mathematical writers have considered it best to publish their works in German. With the increase of scientific interest in China and Japan, where English is used much more than German or French, and with the growth of scientific interest among English speaking nations, it would appear that the English scientific works should soon command the most extensive market, and that more enterprising publishers of English scientific literature should come into existence.

There is a large number of other good general works on mathematics but the few that have been mentioned are among the very best, and they contain references to a large number of others. If any one should desire a cheaper guide through the mathematical literature but one which is fairly reliable and fairly extensive, he might be pleased with Mueller's *Fuehrer durch die mathematische Literature*, published by Teubner in 1909, and costing about two dollars.

In a very general way one may say that the German language contains the most along the lines of general literature, including the history of mathematics, as well as along algebraic and number theory lines. The French is richest along the line of analysis and comprehensive treatises on extensive fields of mathematics. The Italians are now ahead along geometrical lines, while the English excel in mathematical physics. It is hoped that the rapid recent advances in good English literature on various fields of pure

mathematics will have a good effect in creating a greater demand for such literature. Two recent American publications which may reasonably be expected to exert a good influence along this line are J. W. Young's *Fundamental Concepts of Algebra and Geometry,* and J. W. A. Young's *Monographs on Modern Mathematics.* These were published by the Macmillan Company, and by Longmans, Green and Company, respectively.

NOTE ON THE BINOMIAL SERIES.

By K. OGURA in Sendai, Japan.

Euler proved the binomial theorem

$$F(y)=(1+x)^y=\sum_{n=0}^{\infty}\frac{y(y-1)\dots(y-n+1)}{n!}x^n \qquad |x|<1,$$

provided that y is a rational number. There are various proofs of the above formula when y is irrational.

But it is hoped that the following proof will be found both simple and rigorous.

In the binomial series

$$F(y)=\sum_{n=0}^{\infty}\frac{y(y-1)\dots(y-n+1)}{n.}x^n \qquad |x|<1,$$

put

$$f_n(y)=\frac{y(y-1)\dots(y-n+1)}{1.2\dots n}.$$

Let us denote by G a finite positive number however great, and let us take y so that $-G<y<G$.

(i) If $y>-1$, we can choose the positive numbers k and n so that $y+1<k<n$. Then

$$0<\frac{y+1}{k}<1, \quad 0<\frac{y+1}{k+1}<1, \dots, 0<\frac{y+1}{n}<1, \dots$$

and

$$|f_n(y)|=\left|(1-\frac{y+1}{1})(1-\frac{y+1}{2})\dots(1-\frac{y+1}{k-1})\right|\times(1-\frac{y+1}{k})\dots(1-\frac{y+1}{n}).$$

But since

$$\lim_{n=\infty} \sum_{\nu=1}^{n} \frac{y+1}{\nu} = +\infty,$$

we have

$$\prod_{n=k}^{\infty} \left(1 - \frac{y+1}{n}\right) = 0.$$

Hence

$$\lim_{n=\infty} |f_n(y)| = 0.$$

Therefore we can choose n so that $|f_n(y)| < A$, A being a fixed positive number.

(ii) If $y=-1$, then

$$|f_n(y)| = 1.$$

(iii) If $y<-1$, we can choose a positive number p, independent of y, so that $-y<1+p<G$. Then

$$|f_n(y)| < (1+\frac{p}{1})(1+\frac{p}{2})\ldots(1+\frac{p}{n}) < e^p . e^{\frac{1}{2}p} \ldots e^{p/n}$$

$$= e^{p(1+\frac{1}{2}+\ldots+1/n)} < e^{p(C+\log n)} = e^{pC} . n^p,$$

C being Euler-Mascheroni's constant.

If we put $A=e^{pC}$, then $|f_n(y)| < A.n^p$.

Hence, in general, there exists a positive number $X=1$ such that

$$|f_n(y)| X^n < An^p$$

for fixed positive A and p.

Therefore, by a theorem due to Pringsheim,* it follows that the series $F(y)$ is a continuous function of y. $(-G<y<G)$.

Now if an irrational number y be represented as the limit of the sequence $(y_1, y_2, \ldots, y_n, \ldots)$ whose terms are all rational, then we have

$$F(y) = \lim_{n=\infty} F(y_n) = \lim_{n=\infty} (1+x)^{y_n}.$$

But by the definition of the power

* Pringsheim, *Ber. München*, 27 (1897), p. 35; Bromwich, *Infinite Series*, 1908, p. 135.

$$\lim_{n=\infty} (1+x)^{y_n}=(1+x)^y.$$

Hence we have $F(y)=(1+x)^y$; that is,

$$(1+x)^y=1+\frac{y}{1\,!}x+\frac{y(y-1)}{2!}x^2+\ldots \quad |\,x\,|<1.$$

Thus the binomial theorem is proved for all finite values of y.

MOMENT OF INERTIA OF A RING CALCULATED BY AN ELEMENTARY METHOD.

By B. H. BROWN, Whitman College, Walla Walla, Washington.

The problem of finding the moment of inertia of a homogeneous circular ring, of unit density and of radius R with cross-section πr^2, when referred to a diameter, appears to be quite generally avoided by our writers on Elementary Mechanics. The following solution may prove of interest.

Suppose the ring to be edgewise to the plane of the paper and bisected by the plane of the paper giving the two circles of the figure as cross-sections of the ring. Plane PQ perpendicular to the paper bisects the ring and contains the axis of rotation, PQ passing through O, the center of the ring. The moment of inertia of the central section containing the plane PQ and of thickness dy is given by

$$I_c=\frac{\pi}{4}(A^4-a^4)dy. \quad \text{But } A=R+(r^2-y^2)^{\frac{1}{2}} \text{ and } a=R-(r^2-y^2)^{\frac{1}{2}}.$$

$$\therefore I_c=\frac{\pi}{4}\{[R+(r^2-y^2)^{\frac{1}{2}}]^4-[R-(r^2-y^2)^{\frac{1}{2}}]^4\}dy$$

$$=2\pi R^3(r^2-y^2)^{\frac{1}{2}}dy+2\pi R(r^2-y^2)^{\frac{3}{2}}dy.$$

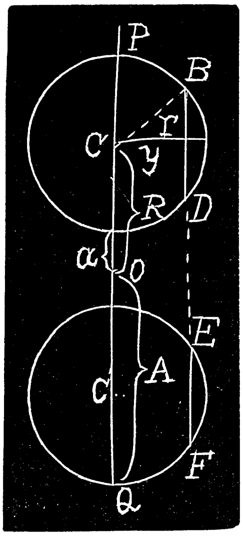
The moment of inertia for any other section as BF parallel to PQ at a distance y from the axis is of course given by $I_y=I_c+my^2$. .

The area $BDEF$ cut from the ring by plane BF is equal to

$$\pi[R+(r^2-y^2)^{\frac{1}{2}}]^2-\pi[R-(r^2-y^2)^{\frac{1}{2}}]^2 \text{ or } 4\pi R(r^2-y^2)^{\frac{1}{2}},$$

and m equals this quantity multiplied by dy. Then

$$I_y = I_c + my^2 = 2\pi R^3 (r^2 - y^2)^{\frac{1}{2}}\, dy$$
$$+ 2\pi R (r^2 - y^2)^{\frac{3}{2}}\, dy$$
$$+ 4\pi R y^2 (r^2 - y^2)^{\frac{1}{2}}\, dy.$$



For the whole ring the formula becomes

$$I_w = \int_{-r}^{+r} [2\pi R^3 (r^2 - y^2)^{\frac{1}{2}} + 2\pi R (r^2 - y^2)^{\frac{3}{2}}$$
$$+ 4\pi R y^2 (r^2 - y^2)^{\frac{1}{2}}]\, dy = \pi^2 r^2 R [R^2 + \tfrac{5}{4} r^2] \ldots (1).$$

Since $2\pi^2 r^2 R$ = mass of ring, M, $I_w = \frac{M}{2}[R^2 + \frac{5}{4} r^2]$.

From (1), when $R = 0$, $I_w = 0$, the reversed moment of inertia of the inner portion of the ring exactly balancing that of the outer portion as the inner part backs over the center and becomes equal in mass to the outer part. This interesting hypothetical process by which the ring converts itself into two coinciding spheres with annulment of moment of inertia may perhaps be better appreciated by finding the moments of inertia of the outer and inner parts separately.

Take as the outer portion of the ring the part outside a cylindrical shell of radius R, with axis at O and perpendicular to the plane PQ. The inner part referred to below will be within this cylinder.

As before, $I_0 = \frac{\pi}{4}(A_1^4 - a_1^4)\, dy$, only in this case $a = R$.

$$\therefore I_o = \frac{\pi}{4}\{[R + (r^2 - y^2)^{\frac{1}{2}}]^4 - R^4\}\, dy.$$

As before, after making obvious changes,

$$my^2 = \pi y^2 [2R(r^2 - y^2)^{\frac{1}{2}} + r^2 - y^2]\, dy.$$

Then

$$I_{O_1} = I_0 + my^2 = \frac{\pi}{4}\int_{-r}^{+r} [4R^3 (r^2 - y^2)^{\frac{1}{2}} + 6R^2 (r^2 - y^2) + 4R^2 (r^2 - y^2)^{\frac{3}{2}}$$

$$+ r^4 - 2r^2 y^2 + y^4]\, dy + \pi \int_{-r}^{+r} [2R(r^2 - y^2)^{\frac{1}{2}} + r^2 - y^2] y^2\, dy.$$

$$\therefore I_{O_1}=\frac{\pi}{2}[\pi r^2R^3+\tfrac{5}{4}\pi r^4R+4r^3R^2+\tfrac{16}{15}r^5]$$

$$=\frac{\pi^2r^2R}{2}[R^2+\tfrac{5}{4}r^2]+[2\pi r^3R^2+\tfrac{8}{15}\pi r^5]\ldots(2).$$

It may be noted that (2) is equal to one-half the moment of inertia of the entire ring, plus the quantity $2\pi r^3R^2+\frac{8}{15}\pi r^5$. When, in this case, $R=0$, the outer portion evidently becomes a sphere of radius r, and I_{O_1} becomes $\frac{8}{15}\pi r^5$, or $\frac{2}{5}Mr^2$, where M is the mass of the resulting sphere. This, of course, is as it should be.

Similar computations for the inner portion of the ring give:

$$I_i=\frac{\pi^2r^2R}{2}[R^2+\tfrac{5}{4}r^2]-[2\pi r^3R^2+\tfrac{8}{15}\pi r^5]\ldots(3).$$

When $R=0$, the inner portion of the ring backs across the center, "turning wrong side out" in becoming a sphere, and its moment of inertia is $-\frac{8}{15}\pi r^5$, or $-\frac{2}{5}Mr^2$. It may be noticed that, when $R=0$, the value of I for the inner portion of the ring is the same as that for the outer portion, but with its sign changed. On comparing (2) and (3), it is seen that their sum is equal to (1) for $R>r$ and $R=r$, and that their sum is equal to zero when $R=0$. Between $R=r$ and $R=0$, equation (1) passes through a series of complications which are interesting, but perhaps more so to the mathematician than to the experimental physicist.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

365. Proposed by C. N. SCHMALL, New York City.

In still water, a steam tug goes 6 miles an hour less when towing a barge than when alone. Having drawn the barge 30 miles up a stream, whose current runs 1 mile an hour, it returns alone and completes the journey in 12 8/11 hours. Find the rate of the tug in still water.

Solution by MARY INGRAM, Lucus (Kansas) High School.

Let x=rate of the tug in still water; $x-7$=rate up stream with barge; $x+1$=rate down stream.

$\frac{30}{x-7}$=time up stream with barge; $\frac{30}{x+1}$=time in returning down stream alone.

$$\frac{30}{x-7}+\frac{30}{x+1}=12\tfrac{8}{11}=\tfrac{140}{11},$$

or, clearing of fractions, $330x+330+330x-2310=140x^2-840x-980$.

Whence, $7x^2-75x+50=0$, and therefore $7x=70$ or 5.

$x=10$ or $\frac{5}{7}$. $\frac{5}{7}$ is not admissible.

Also solved by **M. A. Muzzy, A. H. Holmes, J. Scheffer,** and **H. C. Feemster.**

366. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics and Astronomy, Ohio University Athens, Ohio.

Eliminate m from the equations

$$3a^2m^4+4aym^3+6axm^2-(x^2+y^2-4ax)=0;$$
$$(x^2+y^2-4ax)m^4-6axm^2+4aym-3a^2=0.$$

Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Dividing the first equation by m^4, and adding, we get

$$4ay(m+\frac{1}{m})-6ax(m^2-\frac{1}{m^2})+(x^2+y^2-4ax)\ (m^4-\frac{1}{m^4})$$

and dividing this by $m+\frac{1}{m}$, we have

$$4ay-6ax(m-\frac{1}{m})+(x^2+y^2-4ax)\ (m^2+\frac{1}{m^2})\ (m-\frac{1}{m})=0.$$

Putting $m-\frac{1}{m}=p$, the last expression reduces to

$$4ay-6axp+(x^2+y^2-4ax)(p^2+2)p=0\ldots\text{(I)}.$$

Again dividing both of the original equations by m^2 and adding, we have

$$3a^2(m^2-\frac{1}{m^2})+4ay(m+\frac{1}{m})+(x^2+y^2-4ax)\ (m^2-\frac{1}{m^2})=0.$$

Suppressing the factor $m+\frac{1}{m}$, we get

$$3a^2\left(m-\frac{1}{m}\right)+4ay+(x^2+y^2-4ax)\left(m-\frac{1}{m}\right)=0;\text{ or}$$

$$3a^2p+4ay+(x^2+y^2-4ax)p=0\ldots\text{(II)}.$$

From (II), we get $p=-\dfrac{4ay}{x^2+y^2-4ax+3a^2}$, and substituting this in (I), we get

$$(x^2+y^2+3a^2-4ax)^2(3a^2+10ax-x^2-y^2)-16a^2y^2(x^2+y^2-4ax)=0.$$

Also solved by M. A. Harding, S. Lefschetz, and A. H. Holmes.

367. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Michigan.

Solve the simultaneous equations:

$$\frac{2x}{1+x^2}=y\ldots(1);\quad \frac{2y}{1+y^2}=z\ldots(2);\quad \frac{2z}{1+z^2}=u\ldots(3);\quad \frac{2u}{1+u^2}=x\ldots(4).$$

Solution by PROFESSOR F. L. GRIFFIN, Reed College, Portland, Oregon.

By inspection three solutions are $x=y=z=u=1$, or -1, or 0; and there can be but fourteen others. Now let $x=i\tan\theta$, $[i=\sqrt{-1}]$, whence $y=i\tan 2\theta$, $z=i\tan 4\theta$, $u=i\tan 8\theta$, and $x=i\tan 16\theta$. But $\tan 16\theta=\tan\theta$ for finite values of θ only if $16\theta=\theta+n\pi$, or $\theta=n\pi/15$. Thus we have

$$\begin{array}{llllll}
x=0, & i\tan\pi/15, & i\tan 2\pi/15, & i\tan 3\pi/15, & \ldots, & i\tan 14\pi/15; \\
y=0, & i\tan 2\pi/15, & i\tan 4\pi/15, & i\tan 6\pi/15, & \ldots, & i\tan 28\pi/15; \\
z=0, & i\tan 4\pi/15, & i\tan 8\pi/15, & i\tan 12\pi/15, & \ldots, & i\tan 56\pi/15; \\
u=0, & i\tan 8\pi/15, & i\tan 16\pi/15, & \ldots & \ldots & \ldots
\end{array}$$

the values for y, z, u being of course the same sets as for x in different orders. The values $x=+1$, -1 correspond to infinite values of θ for which $\tan\theta=-i$, $+i$.

GEOMETRY.

393. Proposed by S. LEFSEHETZ, University of Nebraska.

Draw a triangle having a given angle, and with its vertices on three given concentric circles.

I. Solution by H. C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Nebraska.

Let $x^2+y^2=c^2$, $x^2+y^2=b^2$, $x^2+y^2=a^2$, be the given circles, and ϕ the given angle. Place the vertex of the given angle, ϕ, on the circumference of the first circle at 0, c, the angle being formed by the lines,

$$y=lx+c, \text{ and } y=\frac{l+\tan\phi}{1-l\tan\phi}x+c,$$

where l is the tangent of the angle made by the first line with the x-axis, in the usual sense.

Substituting the equation of the first line in the equation of the second circle, the condition for intersection is

$$l \overline{\overline{>}} \frac{\surd(c^2-b^2)}{b}, \text{ or } l \overline{\overline{<}} -\frac{\surd(c^2-b^2)}{b},$$

and substituting the equation of the second line in the equation of the third circle, the condition for intersection is,

$$\frac{l+\tan\phi}{1-l\tan\phi} \overline{\overline{>}} \frac{\surd(c^2-a^2)}{a}, \text{ or } \frac{l+\tan\phi}{1-l\tan\phi} \overline{\overline{<}} -\frac{\surd(c^2-a^2)}{a},$$

which reduces to

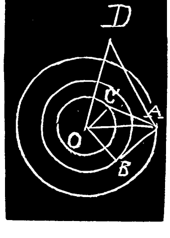
$$l \overline{\overline{>}} \frac{a\tan\phi-\surd(c^2-a^2)}{\surd(c^2-a^2)\tan\phi+a}, \text{ or } l \overline{\overline{<}} \frac{a\tan\phi+\surd(c^2-a^2)}{a-\surd(c^2-a^2)\tan\phi}.$$

Taking $c>b$ and $c>a$, there is no solution if

$$\phi>\tan^{-1}\frac{b\surd(c^2-a^2)+a\surd(c^2-b^2)}{\surd(c^2-a^2)\surd(c^2-b^2)-ab},$$

with the vertex of ϕ on the circumference of the first circle. But any angle gives a solution by placing its vertex on the circumference of either of the other circles, by solving in a similar manner. With these exceptions, any value of l, except the limiting value, which gives only one solution, gives four solutions.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.



With one angle given, there is an indefinite number of triangles, and consequently I will modify the problem to having the three angles given; or, in other words, to describe a triangle whose vertices are in the circumference of three given concentric circles similar to a given triangle.

On the radius OA draw the triangle OAD similar to the given one; draw DC from $OA:OB=AD:DC$. Make $\angle AOB=\angle ADC$, connect B with C, then ABC will be the

required triangle. For, by construction, $\triangle ABC$ is similar to $\triangle AOB$, and, consequently, $\triangle ABC$, the vertices of which lie on the three circumferences, is similar to $\triangle OAD$.

Solved similarly by C. N. Schmall.

394. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

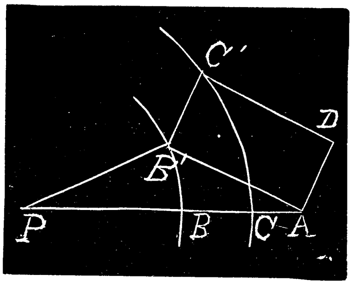
The joins of the excentres to the corresponding vertices of the pedal triangle are concurrent.

No solution of this problem has been received.

395. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

From a point P without a rectangular field ABC the distances PA, PB, PC measured to the corners are respectively 70, 40, 60 chains. What is the area of the field?

Solution by A. H. HOLMES, Brunswick, Maine; J. SCHEFFER, A. M., Hagerstown, Maryland, and M. A. MUZZY, Waterville, Maine.



Let P be the point without the field. Draw the line PBA; on it lay off $PB=40$ chains, $PC=60$ chains, and $PA=70$ chains. Then any point on the circumference of circle whose center is P and radius PB may be one corner of the rectangular field. One such field is $AB'C'D$. Hence, the problem is indeterminate. To make the problem determinate, one of the angles APB', APC', or $B'PC'$ should be given.

396. Proposed by DANIEL KRETH, Oxford, Iowa.

In the triangle ABC, $AB=214$, $BC=263$, and $AC=405$. A point P is situated in the same horizontal plane; angle $BPA=13°\ 30'$ and angle $BPC=29°\ 50'$. Find the distances, AP, BP, and CP.

Solution by M. A. MUZZY, Waterville, Maine.

Draw a circle through P, A, C, cutting BP in some point D.* Then $BPA=ACD$, $BPC=CAD$, or $BPC=180°-CAD$. Hence $CAD=29°\ 50'$ or $150°\ 10'$, $ADC=136°\ 40'$ or $16°\ 20$. There are four solutions.

In triangles CAD find $CD_1=CD_2=293.60$, $CD_3=CD_4=716.43$.

In triangle ABC find $ACB=28°\ 23.8'$. $BCD=ACB\mp ACD$. Whence $BCD_1=BCD_4=14°\ 53.8'$, $BCD_2=BCD_3=41°\ 53.8'$.

In triangles BCD find $CBD_1=105°\ 21.2'$, $CBD_2=77°\ 13.4'$, $CBD_3=119°\ 27.8'$, $CBD_4=156°\ 46.0'$.

In triangles CBP find $BP_1=372.59$, $BP_2=389.09$, $BP_3=269.93$, $BP_4=422.58$; $CP_1=509.80$, $CP_2=515.58$, $CP_3=460.29$, $CP_4=208.55$.

In triangles APB find $AP_1=166.77$, $AP_2=572.10$, $AP_3=57.96$, $AP_4=600.81$.

Also solved by J. Scheffer and H. C. Feemster.

CALCULUS.

320. Proposed by J. F. LAWRENCE, Stillwater, Oklahoma.

Show that, if $u=1+A_1x+\frac{1}{2\,!}A_2x^2+\frac{1}{3\,!}A_3x^3+\ldots$ where the quantities A are connected by the relation $A_m=mA_{m-1}-\frac{1}{2}(m-1)(m-2)A_{m-3}$, then $\log[u(1-x)^{\frac{1}{2}}]=\frac{1}{2}x+\frac{1}{4}x^2$. [From Forsyth's *Differential Equations*, p. 48.]

I. Solution by E. B. ESCOTT, Ann Arbor, Michigan.

$$u=1+A_1x+\frac{1}{2\,!}A_2x^2+\ldots+\frac{A_m}{m\,!}x^m+\ldots$$

where $A_m=mA_{m-1}-\frac{1}{2}(m-1)(m-2)A_{m-3}$.

Form the expression which has for the coefficient of x^m,

$$A_m-mA_{m-1}+\tfrac{1}{2}(m-1)(m-2)A_{m-3}$$

multiplied by some factor. Then such an expression will have only a finite number of terms. Such an expression is

$$u-xu+\tfrac{1}{2}\int x^2u\,dx;$$

also its derivative

$$(1-x)\frac{du}{dx}+(\tfrac{1}{2}x^2-1)u.$$

Therefore, we have for the differential equation of u,

$$(1-x)\frac{du}{dx}+(\tfrac{1}{2}x^2-1)u=0.$$

Separating variables,

$$\frac{du}{u}+\frac{\frac{1}{2}x^2-1}{1-x}dx=0.$$

whence

$$\log u(1-x)^{\frac{1}{2}}=\frac{x}{2}+\frac{x^2}{4},$$

the constant of integration being zero, since when $x=0$, $u=1$, and $\log 1=0$.

II. Solution by A. M. HARDING, University of Arkansas, Fayetteville, Arkansas.

From the given relation we have

$$A_1=1,\ A_2=2,\ \ldots,\ \tfrac{1}{2}\frac{A_{m-3}}{(m-3)!}=\frac{mA_{m-1}}{(m-1)!}-\frac{mA_m}{m!}\ldots(1).$$

Let $\phi(x)=(1-\frac{x^2}{2})(1+A_1x+\frac{A_2}{2!}x^2+\ldots+\frac{1}{m!}x^m+\ldots)(1+x+x^2+\ldots+x^m+\ldots)$

$$=[1+A_1x+(\frac{A_2}{2!}-1)x^2+\ldots+(\frac{A_m}{m!}-\tfrac{1}{2}\frac{A_{m-2}}{(m-2)!})x^m+\ldots]\times$$

$$[1+x+x^2+\ldots+x^m+\ldots]$$

Let us assume that both these series are absolutely convergent. Multiply them in the ordinary way and obtain a series $\Sigma c_m x^m$ where c_m, after reduction, has the value

$$c_m=\frac{A_m}{m!}+\frac{A_{m-1}}{(m-1)!}+\tfrac{1}{2}\left[\frac{A_{m-2}}{(m-2)!}+\frac{A_{m-3}}{(m-3)!}+\ldots+\frac{A_2}{2!}+A_1+1\right].$$

From the relation (1) we have

$$\left\{\begin{array}{l}
\tfrac{1}{2}=\frac{3A_2}{2!}-\frac{3A_3}{3!}=1+A_1+\frac{A_2}{2!}-\frac{3A_3}{3!}\quad (\text{since } \frac{3A_2}{2!}=1+A_1+\frac{A_2}{2!}=3. \\
\tfrac{1}{2}A_1=\frac{4A_3}{3!}-\frac{4A_4}{4!}. \\
\tfrac{1}{2}\frac{A_2}{2!}=\frac{5A_4}{4!}-\frac{5A_5}{5!}. \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\tfrac{1}{2}\frac{A_{m-2}}{(m-2)!}=(m+1)\frac{A_m}{m!}-(m+1)\frac{A_{m+1}}{(m+1)!}.
\end{array}\right.$$

Adding these equations, we obtain

$$0=c_m-(m+1)\frac{A_{m+1}}{(m+1)!}.$$

$$\therefore c_m=(m+1)\frac{A_{m+1}}{(m+1)!}\text{ and }\phi(x)=\Sigma c_m x^m=A_1+A_2x+\frac{1}{2\,!}A_3x^2+\ldots$$

$$+\frac{1}{m!}A_{m+1}x^m.$$

$$\therefore \phi(x)=\frac{du}{u}. \quad \text{But } \phi(x)\equiv(1-\frac{x^2}{2})(1-x)^{-1}.u=\frac{u}{2}[1+x+\frac{1}{1-x}].$$

$$\therefore \frac{du}{u}-\frac{dx}{2(1-x)}=(\tfrac{1}{2}+\frac{x}{2})dx.$$ Integrating, we obtain

$$\log u+\tfrac{1}{2}\log(1-x)=\frac{x}{2}+\frac{x^2}{4}+c.$$

Now when $x=0$, $u=1$, and hence $c=0$.

$$\therefore \log[u(1-x)^{\frac{1}{2}}]=\frac{x}{2}+\frac{x^2}{4}.$$

C. N. Schmall should have received credit for solving 317.

MECHANICS.

357. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

A portion of a circular cylinder cut off by two planes through the axis rests with its curved surface on two rough horizontal rails parallel to its axis, the coefficients of friction μ_1, μ_2 at upper and lower rails respectively. If the body is in limiting equilibrium at both rails when the plane through the axis and the center of gravity is perpendicular to both rails, find the distance of the center of gravity in terms of the distance betwen the rails, the inclination of their plane to the horizon, and the cofficients of friction.

358. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Two heavy particles connected by a string, length l, lie one on each of two inclined planes with common horizontal edge and of angles α and β. The inclination of the string to the edge varies as the inclination to the horizon of a simple pendulum of length $l(\sin\alpha+\sin\beta)$.

No solutions of these problems have been received.

259. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A uniform beam of the weight W, rests on a horizontal plane, and leans against a vertical wall, but so as *not* to lie in a vertical plane. Denoting the pressure upon the horizontal and vertical planes, respectively, by x

with AB before and after impact; u and v the components of the velocities along AB before and after impact. The tangential components of these velocities (along QP) before and after impact are $u\tan\theta$ and $v\tan\phi$, which are unchanged by the impact, since the contact is smooth. Hence it must be that $u\tan\theta=v\tan\phi$, or since $v=eu$, $\tan\theta=e\tan\phi$. Also since the string tangent to the rim is a normal to its involute, $\theta+\phi=90^\circ$ and $\tan\phi=\cot\theta$. Hence $\tan\theta=\sqrt{e}$. But $\theta=PQR$ and $POR=90^\circ+\theta$. Therefore the required condition is $l=a(\pi/2+\tan^{-1}\sqrt{e}+1/\sqrt{e})$.

Also solved by S. G. Barton.

PROBLEMS FOR SOLUTION.

ALGEBRA.

374. Proposed by H. PRIME, Boston, Massachusetts.

Divide an angle of 30° into two parts so that the product of the third and fourth powers of their sines (or cosines) shall be a maximum. To be solved without using the methods of calculus. [From *The Maine Farmers' Almanac*, 1912.]

375. Proposed by S. LEFSCHETZ, University of Nebraska.

Prove that $\dfrac{e}{2m+2}<e-(1+\dfrac{1}{m})^m<\dfrac{e}{2m+1}$. [Schlömilch.]

376. Proposed by W. W. BEMAN, Professor of Mathematics, University of Michigan, Ann Arbor, Michigan.

If $\dfrac{(1+1/m)^m}{e}=1-a_1\dfrac{1}{m}+a_2\dfrac{1}{m^2}-a_3\dfrac{1}{m^3}+\ldots$, prove $na_n=\sum_{k=1}^{k=n}\dfrac{k}{k+1}a_n-k$, and compute a_1, a_2, a_3, ..., a_n.

GEOMETRY.

403. Proposed by C. E. GITHENS, Wheeling, West Virginia.

In a triangular field the sides enclosing an obtuse angle are 35 rods and 48 rods in length, Two straight lines are drawn from this vertex, and are at right angles to these sides. If these lines intersect the base 16 rods apart, how long is the third side of the field?

404. Proposed by S. LEFSCHETZ, Ph. D., University of Nebraska.

Let ABC be a triangle, M a point on BC, BE the intersection of the perpendicular to AM in M with AB and AC, F, the other intersection of the circle circumscribed to ABC with the circle through M and A orthogonal to it in A. Prove that the points C, D, E, F are on a circle, and find the envelope of the latter when M describes BC.

405. Proposed by C. N. SCHMALL, New York City.

A plane cuts a constant volume from a given right cone. Prove that the minor axis of the section has a constant length.

CALCULUS.

325. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Michigan.

Integrate $\int \frac{4x^6-a^3}{\sqrt{(x^6-a^3)}}dx$.

326. Proposed by C. N. SCHMALL, New York City.

Prove $\int_0^\infty \frac{[(\tan^{-1}ax)^2-(\tan^{-1}bx)^2]dx}{x}=\frac{1}{4}\pi^2(\log a-\log b)$.

327. Proposed by RICHARD P. LOCHNER, 214 North 63rd Street, Philadelphia, Pennsylvania.

A hound is at the middle point of the side of a square field, and a fox is at an adjacent corner. How far will the hound run to catch the fox if the fox runs on the perimeter of the field and the hound runs directly towards the fox at all times, the hound running n times as fast as the fox? Where will the race end?

328. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Prove that, $\frac{E_m}{\pi/2m}\int_0^{\pi/2m}[\sin\alpha+\sin(\frac{\pi}{m}+\alpha)+\ldots\sin(\pi\frac{m-1}{m}+\alpha)]d\alpha=\frac{2mE_m}{\pi}$.

[Sheldon and Hausmann: *Dynamo Electric Machinery*, Vol. I, p. 51.]

MECHANICS.

269. Proposed by F. H. SAFFORD, Ph. D., University of Pennsylvania.

Three jointed rods, lengths 5, 2, and 5 units, are suspended from two pins which are 6 units apart and on a horizontal line, forming a freely jointed quadrilateral with the shortest side at the bottom. At the two lower joints are weights of A and $2A$, respectively. Find the position of rest, the reactions along each bar, and the pressures on the pins.

270. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Burghfield, England.

A cycloid has its base vertical. Find the line of quickest descent from the middle point of the base, and its approximate inclination to the horizon.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

186. Proposed by H. PRIME, Boston, Massachusetts.

Show that $\frac{(n+1)(n+2)\ldots(2n-2)}{(n-1)!}$ is an integer for all values of n.

NOTES AND NEWS.

It is now hoped that the printing of the large Italian *Encyclopedia of Elementary Mathematics* will begin during the present year. The work is to be published by Hoepli of Milan. M.

The next meeting of the National Education Association will be held in Chicago, June 6-12, 1912. This will be the first time that this body has met in Chicago in twenty-five years. S.

The trustees of the University of Chicago have announced a system of retiring allowances which is to go into effect at once and for which endowment funds of more than two million dollars are to be set aside. S.

The ninth annual meeting of Ohio teachers of mathematics and science was held in Columbus on Friday and Saturday, March 29-30, 1912. Among the papers on mathematics was one on the teaching of trigonometry by Professor B. F. Yanney of Wooster University. S.

The next regular meeting of the Chicago Section of the American Mathematical Society will be held at Cleveland, Ohio, in affiliation with the American Association for the Advancement of Science. The latter will open its meeting on December 30, 1912. M.

According to the latest Annual Register of the American Mathematical Society about 50 of the 668 members are women. It is interesting to observe that the American Mathematical Society has a much larger per cent. of women members than the leading mathematical societies of Europe. According to the latest register of the German mathematical society (*Deutschen Mathematik Vereinigung*) only 5 of its 759 members are women; and only one of these 5 members is a German woman, while three of them are Americans and the remaining one is a Russian. The French mathematical society has also very few women members. The numbers of the women members of the Circolo Mathematico di Palermo and of the London Mathematical Society are considerably larger but they are much smaller than in our own society. M.

The February, 1912, number of the *Bulletin of the American Mathematical Society* contained a circular of 12 pages describing the French edition of the large mathematical encyclopedia. According to this circular 12 parts of the encyclopedia, covering 2762 pages, had appeared and 25 parts were then in press. A list of 160 contributors, including the names of such prominent French mathematicians as Poincaré, Picard, and Jordan, is given in this circular. More than one-third of the names appearing on this list are those of mathematicians residing in Paris, which gives some indication of the centralization of French Mathematics. It is also significant that this list contains the names of four Italian mathematicians. The circular is dated December, 1911, and it was issued by Gauthier-Villars of Paris, France. M.

C. F. GAUSS AND HIS SONS — A CORRECTION. In the January number of the MONTHLY, page 2, occurs the statement that John Bolyai was "a victim of the meanness of Gauss, as was also his own son who passed his life an exile here in Colorado."

My friend, Professor Cajori, informs me that no one of Gauss's four sons made his home in Colorado. He tells me that the son Eugen had a quarrel with his father, because of Eugen's wild life at the University of Göttingen, but when Eugen left home to sail for America, his father followed him, urged him to return home, and, when failing in this effort, offered him money to take with him to America. Eugen left home and came to this country by his own choice and against the wishes of his parents. His father was greatly grieved, as is shown by his correspondence. See the article "Carl Friedrich Gauss and his Children," in *Science*, N. S. Vol. 9, 1899, pages 697-704. G. B. HALSTED.

The thirtieth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 5 and 6, 1912. The total attendance upon the four half-day sessions was seventy-five, including fifty-six members of the Society.

There were twenty-nine papers presented, as follows:

(1) On birational transformations of three-space related to four-space varieties. (15 minutes.) Dr. S. Lefschetz, University of Nebraska.

(2) Optical interpretations in higher geodesy. (10 minutes.) Professor W. H. Roever, Washington University.

(3) Mechanisms for illustrating lines of force. (10 minutes.) Professor W. H. Roever.

(4) Deviations of falling bodies for a distribution not of revolution. Second paper. (15 minutes.) Professor W. H. Roever.

(5) The geometry of conformal rational transformations in a plane. (20 minutes.) Professor Arnold Emch, University of Illinois.

(6) The present state of the theory of Jupiter's five minor satellites. (20 minutes.) Professor Kurt Laves, University of Chicago.

(7) An extension of Descarte's rule of signs. Second paper. (20 minutes.) Professor D. R. Curtiss, Northwestern University.

(8) Equality in geometry. (15 minutes.) Professor J. K. Whittemore, Western Reserve University.

(9) On the relation between the empirical and the inertial trihedrons of gravitational astronomy. (20 minutes.) Dr. G. O. James, Washington University.

(10) A forgotten theorem of Newton's on planetary motion and an instrumental solution of Kepler's equation. (10 minutes.) Professor E. J. Wilczynski, University of Chicago.

(11) The relative number field $K(\sqrt[l]{a})$. (15 minutes.) Dr. G. E. Wahlin, University of Illinois.

(12) Analytic curves in noneuclidean space. Second paper. (20 minutes.) Dr. E. G. Bill, Purdue University.

(13) Algebra in the Quadripartitum numerorum of Johannes de Muris. (10 minutes.) Professor L. C. Karpinski, University of Michigan.

(14) Infinite systems of indivisible groups. (15 minutes.) Professor G. A. Miller, University of Illinois.

(15) Projective differential geometry of developable surfaces. (10 minutes.) Mr. W. W. Denton, University of Illinois.

(16) The method of monodromie and its application to three-parameter quartics. (10 minutes.) Professor R. P. Baker, University of Iowa.

(17) On Transcendentally transcendental functions. (25 minutes.) Professor R. D. Carmichael, Indiana University.

(18) On the theory of linear difference equations. (20 minutes.) Professor R. D. Carmichael.

(19) A general formula for the valuation of securities. (15 minutes.) Professor J. W. Glover, University of Michigan.

(20) On ordinary plane and skew curves. (By title.) Dr. E. L. Dodd, University of Texas.

(21) On the Spanish symbol U for thousands. (By title.) Professor Florian Cajori, Colorado College.

(22) Infinite developments and the composition property $(K_{1,2}\,B_1)_*$ in general analysis. Mr. E. W. Chittenden, University of Chicago.

(23) Note on Mersenne's numbers. Mr. V. M. Spunar, Chicago, Ill.

(24) The projective differential geometry of three-spreads generated by ∞^1 planes in five dimensional space. Dr. E. B. Stouffer, University of Illinois.

(25) Multiplicative interrelations of certain classes of sequences of positive terms. Professor E. H. Moore, University of Chicago.

(26) On a theorem of Fejer's and an analogons to Gibbs' phenomenon. Dr. T. H. Gronwall, Chicago, Ill.

(27) Some asymptotic expressions in the theory of numbers. Dr. Gronwall.

(28) Closed orbits of ejection and related periodic orbits in the problem of three bodies. Professor F. R. Moulton, University of Chicago.

(29) Necessary and sufficient conditions for the interchange of limit and summation for a special type of series. Dr. T. H. Hildebrandt, University of Michigan.

An interesting feature of this meeting was the provision in the program for "notes and queries" on any topics not directly related to any of the papers presented. This brought out some very interesting informal discussion.

S.

BOOKS.

A Treatise on the Analytical Geometry in Three Dimensions. By George Salmon, D. D., D. C. L., LL. D., F. R. S., late Provost of Trinity College. Fifth edition. Revised by Reginald A. P. Rogers, Fellow of Trinity College, Dublin. Vol. I. 8vo, cloth. xxii+470 pages. Price, $3.00. New York: Longmans, Green & Co.

In the revision of this, the most exhaustive and noted work of its kind in the English language, it is the purpose of Mr. Rogers to preserve the substance of the fourth edition. Copies of the fourth and last edition had become quite difficult to obtain. In this new edition, Mr. Rogers has added some new mattter of much value. For example, in Chapter V, we find excellent illustrations of models of the different species of quadrics, articles or paragraphs on the analytical classification of real quadrics (88_a), and elsewhere, on projection and Fiedler's projective coördinates (144_c), on non-Euclidean theory of distance and angle (144_a), and on the expression of twisted cubics and quartics by rational or elliptic parameters, (333_a, 347_a, 348, 349).

In differential geometry, the aim has been to form a closer connecting link between Dr. Salmon's book and the more extensive and more purely analytical methods used by Bianchi, Darboux, and others. A number of other additions and improvements are made, but space forbids noting any more.

It is the purpose to bring out this fifth edition in two volumes. The fourth edition contained 612 pages. The type in this edition is much superior to that in the fourth edition. It is to be hoped that Volume II will soon be issued. F.

The Elements of Statistical Methods. By Wilford I. King, M. A., Instructor in Statistics in the University of Wisconsin. 8vo, cloth, xvi+250 pages. Price, $1.50. New York: The Macmillan Co.

So far as we know, the text before us is the first of its kind published in America. There have been texts published heretofore on statistical methods in Biology and perhaps other subjects, but none dealing with the method in general.

In the present work, the author has furnished those students of economics, sociology, and others of the educated public unfamiliar with the language and use of mathematics, a simple text easily within their grasp. The work is one that will prove of great value to the student of sociology, economics, as well as to those of some of the natural sciences. F.

Introduction to Analytical Mechanics. By Alex. Ziwet, Professor of Mathematics in the University of Michigan, and Peter Field, Ph. D., Assistant Professor of Mathematics in the University of Michigan. 8vo, cloth, ix+378 pages. Price, $1.60 net. New York: The Macmillan Co.

This volume is intended as a brief introduction to the study of mechanics for Junior and Senior students in colleges and universities. While the work is based largely upon Professor Ziwet's excellent Theoretical Mechanics, yet it differs from it in several essential respects. For example, the applications to engineering are omitted and the analytical treatment has been broadened. Geometrical ideas are made to precede analysis, and in so doing the idea of the vector is freely used while omitting at the same time the special methods and notations of vector analysis.

Those teachers who have used Professor Ziwet's larger work will welcome this briefer course by Ziwet and Field as more suitable for them if they have only a very limited time at their disposal to devote to the subject. F.

The Teaching of Physics for Purposes of General Education. By C. Riborg Mann, Associate Professor of Physics, The University of Chicago. 8vo, cloth, xxv+304 pages. Price, $1.25 net. New York: The Macmillan Co.

This is one of the Teachers' Library Series and the editor, President Butler of Columbia University, in his introduction sets forth eight fundamental principles which should be assumed in the teaching of Physics. Among them are: That the topics chosen and the method pursued should be determined for the intellectual needs and interests of the school; that teacher should put out of his mind the thought that each pupil before him is aiming to become a specialist in physical science; that physical science should not be something fixed and definite; that the student should know something of the men whose names are epoch-making in its history and development; that the ordinary standards for measuring time, space, weight and other characteristics should not be taken for granted; and that accurate measuring, while necessary and important, should hold not first place, but a subordinate place.

The author, Professor Mann, is not only an expert physicist, but he is an excellent teacher as well. He knows just how much help the student needs in making experiments in order to keep up his courage and his interest. His suggestions, therefore, as set forth in this book have a double value since they come from one who is both a teacher and an investigator.

The book is divided into three parts. The first part discusses, The Development of the Present Situation; the second part, Physics and Democratic Education; the third part, Hints at Practical Applications. F.

A Mental Arithmetic for Oral Review. By John Brookie Faught, Ph. D., Professor of Mathematics in the Western State Normal School, Kalamazoo, Mich. Paper back, 24 pages. Published by the author.

It seems that the author of this little work is old-fashioned enough to wish to return to the good old days of the mental arithmetic. What will modern educators think of a man who is now writing a mental arithmetic when our school programs have long passed that stage in educational evolution? Yet we, too, are old-fashioned enough to wish that some of our college Freshmen had some of this mental arithmetic training. Not long ago we witnessed a college Senior in all seriousness add 1/2 and 2/3 and get 3/5 as a result. Now you college teachers of mathematics do not hold your noses while you read that statement, for we 'll warrant that many of you who read it have witnessed examples of the same stolid stupidity. There is something radically wrong somewhere, and to our mind, our critics are failing to put their fingers on the right spot. It seems to us that there is too much of the pyrotechnic display in our scheme of education, too morbid a desire for the novel and spectacular and too little of the serious purposes of life. The average college or university diplomas are coming to have about as much value educationally as a new national bank note would have commercially without the denomination stamp upon it. F.

Linear Polars in the k-Hedron in n-Space. A Dessertation to the Faculty of the Ogden Graduate School of Science, The University of Chicago, in Candidacy for the Degree of Doctor of Philosophy. By Harris Franklin MacNeish.

Copies of this thesis may be secured by sending 25 cents to the University of Chicago Press. F.

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NOTE ON THE DEFINITION OF AN ASYMPTOTE.

By H. L. RIETZ, University of Illinois.

In teaching the subject of asymptotes to a class of sophomore students recently, my attention was called to a point that seemed worth noting, concerning certain definitions of an asymptote to a curve. Text-books in common use differ considerably in their definitions of an asymptote. There are three different definitions that I find by an examination of text-books.

Judging from these books, it seems that a definition often given is the following:

(1) If the tangent to a curve approaches a limiting position, as the distance of the point of contact from the origin is indefinitely increased, this limiting position is called an asymptote. More briefly, it is sometimes stated that an asymptote to a curve is a tangent line whose point of contact is at infinity, but such that the line is not entirely at infinity.

A second definition in rather common use is the following:

(2) An asymptote to a curve is a straight line whose distance from a point on the curve diminishes indefinitely as the point moves along the curve to an infinite distance from the origin.

A third definition sometimes given is:

(3) If a straight line cuts a curve at two points at an infinite distance from the origin, but is not entirely at infinity, the line is called an asymptote.

The book that I am using in the class referred to above, gives the first of these three definitions. It is then shown how to determine the oblique asymptotes of the curve

$$f(x, y)=0, \quad f(x, y) \text{ being a polynomial of degree } n,$$

by selecting m and b in the straight line

$$y=mx+b,$$

so that at least two of the intersections of this line with $f(x, y)=0$ shall be at an infinite distance from the origin.

One of the students criticised the method on the ground that it is not proved thereby, that a line cutting a curve at two points at infinity is a tangent to the curve or the limiting position of a tangent.

To say, as is sometimes done, that it works well to consider the two points of intersections at infinity as coincident, is not very satisfactory to the sophomore mind. No proof that the above method gives the limiting position of a tangent line is carried out, so far as I know, in a beginning course in analytic geometry.

The method that merely determines the parameters m and b in $y=mx+b$ by requiring this line to have two of its intersections with the curve $f(x, y)=0$ at infinity, is a direct application of the third definition, and makes no use of the idea of a tangent. It therefore seems to be a natural method to give the third definition when the above method of finding asymptotes is to be used. The fact that it works well to regard an asymptote thus determined as the limiting position of a tangent, by treating the two points of intersection at infinity as coincident, could be brought to the attention of students very naturally without making the idea of the tangent of such primary importance as to include it in the definition. In this connection, it may well be shown by use of the formula,

$$x=\frac{x_1+rx_2}{1+r},$$

for the division of a line segment, that, for fixed x_1 and x_2, the formula gives one and only one value of x that corresponds to a value of r; and that r can take any value except -1. If $r=-1$, the formula is meaningless, but it is convenient to use the symbol ∞ to correspond to $r=-1$, and to look upon the line of which x_1x_2 is a segment as having only one point at infinity to correspond to $r=-1$.

Such a method seems, to the writer, much more natural than that in which the first definition is employed.

REVIEW OF THE FRENCH EDITION OF HALSTED'S "RATIONAL GEOMETRY."

By JOHN A. EIESLAND, University of West Virginia.

Rational Geometry. By George Bruce Halsted. Second Edition. New York and London: John Wiley and Sons. 1907. Pp. viii+274.

Dr. George-Bruce Halsted. *Géométrie Rationnelle. Traité Elémentaire de la Science de la l'Espace.* Traduction Française par Paul Barbarin. Avec une Préface de C.-A. Laisant. Paris, Gauthier-Villars. 1911. Pp. iv+296.

The French translation by Professor Barbarin of the second edition of Professor Halsted's *Rational Geometry* is a handsome volume published by the book firm of Gauthier-Villars. The book is prefaced by Professor C. A. Laisant, who is well known to American students of mathematics. It is now seven years since the publication of the first edition of this work, a review of which by Professor Davisson of Indiana University was published in the *Bulletin of the American Mathematical Society,* Vol. XI, pp. 330-336.

Poincaré has characterized Halsted's introduction of Hilbert's ideas and principles in an elementary text-book as follows:

"Introduire ce principe dans l'enseignement, c'est bien pour le coup rompre les ponts avec l'intuition sensible, et c'est là je l'avoue, une hardiesse qui me parait presque une témérité."

Daring and rash it may be, but Professor Halsted certainly deserves credit for his temerity, for he has given to American teachers and students of mathematics a well-written text-book on Geometry in which the fruitful ideas and principles of Hilbert have been most ably interpreted and applied. To what extent the book can be used in elementary instruction, as a high-school text-book, for instance, the writer is not prepared to discuss, but it seems reasonable to suppose that it can be taught with the best results to students who have had a preliminary course in geometry such as the one published some time ago by E. Borel,* in which intuitional methods are largely used and formal demonstrations relegated to a second place. The prevalent American custom of introducing a student to formal demonstrative geometry without any preceding intuitional geometry seems pedagogically unwise and is largely responsible for the distaste of the average high-school student for mathematical subjects. The century-long authority of Euclid will of course prevent the book from becoming popular or a good seller. To the average teacher of mathematics with limited training and knowledge of modern mathematics it will seem revolutionary and strange because it is new, not because it is inherently more difficult than Euclid or any of his modern followers. In some respects it is simpler. With Euclid

* E. Borel, *Géométrie, Premier et second cycles.* Paris, Armand Colin, 1905,

the theory of proportion involves the difficult idea of limits and continuity (commensurability); here we have a sect-calculus (Ch. VII) based on very simple assumptions and on Schur's very elementary proof of Pascal's theorem. The vexing question about definitions of point, straight line, plane, is solved by not defining them at all, but assuming them as the "elements" or "things" that form the material of the structure.*

It is not our purpose to review each particular chapter in detail. The changes which have been made in this sécond and revised edition are all decided improvements. The sect-calculus is introduced by Schur's proof of Pascal's theorem on the basis of which the commutative, associative and distributative laws for addition (subtraction), multiplication, and division of sects are proved. This seems preferable to the method of the first edition, which made use of the circle.

The author tries to avoid as much as possible the use of the Archimedes assumption. In the first edition he was disposed to eliminate it altogether, being satisfied with a mere statement of it. However, in Ch. XI on "Length and Content of the Circle" it was found impossible to avoid the use of this assumption and — a consequence of it — the assumption of a unique sect equal to the length of the circle. From the standpoint of rigor this seems satisfactory, although it is not made quite clear in the text how the Archimedes assumption is involved in the assumption: "There is one and only one sect greater than the perimeter of any inscribed polygon and less than the perimeter of any circumscribed polygon, namely, the length of the circle." The real content of the Archimedes assumption, in so far as it applies to this postulate, is thus left as an unexplained mystery. Perhaps this is the only way out of the dilemma; to go into details might also necessitate saying something about the "axiom of completeness" and this would have been decidedly beyond the scope of an elementary book. On the whole we think the author has skilfully avoided the difficulties of this part of the subject without sacrificing rigor.

In writing an elementary text-book based on Hilbert's system of axioms a difficulty presents itself that can only be conquered by evading it. Thus, in the first edition of Hilbert's *Foundations* we find an axiom II, 4 as follows: "Any four points on a straight line can always be so lettered, A, B, C, D, that B is between A and C and also between A and D, and furthermore C is between A and D and also between B and D." This assumption was later found to be a demonstrable theorem,† but we note that

* We note that the "National Committee of Fifteen on Geometric Syllabus" seems to take a similar view: "Certain concepts are so elementary that no simple terms exist by which to define them, although they can easily be explained. For example, *point, line, surface, space, angle, straight line, curve.* The committee recommends that teachers give more attention to instilling a clear concept of such terms and none to exact definitions." This is all very good indeed, but we find on page 30 of the same report among the general list of *postulates:* "A straight line is the shortest line between two points." First the student is told that a straight line is of illimitable extent, and then he is told to believe that such a line is the shortest line. This is truly what Max Dehn calls "Ein energischer widerspruch."

† Proved by R. L. Moore and E. H. Moore.

the proof is rather long and difficult for beginners and the author has therefore rightly placed it in appendix I. The student will readily grant the proposition, appeal being made to intuition, but we have here a striking illustration of the difficulty we meet when any appeal to intuition is denied, as it must be in a Rational Geometry.

And now quite recently in a paper published in the *Annalen*, B. 71, 1911, Rosenthal* has proved that the system of assumptions of congruence can be simplified; he shows that assumption III, 5† is wholly superfluous, the same being true of parts of III, 1 and III, 4. His proofs of these superfluous axioms are not very simple in an elementary sense, and in a future third edition of Halsted's geometry we shall perhaps have an appendix II devoted to these theorems, the proofs of which we hope the author will simplify sufficiently for elementary use. It follows also from Rosenthal's work that the five congruence assumptions for the geometry on the sphere (*Pure Spherics*, Ch. XVI, pp. 226-227, French edition) may be reduced by one and assumptions III, 1 and III, 4 simplified so as to correspond to the same group of assumptions for the plane.

In appendix II the author introduces the basic assumptions for the compasses. Here we find a notable improvement in the present edition, the two assumptions being reduced to one, viz.: "If a straight have a point within a circle, it has two points on the circle." Schur has pointed out that this assumption can be proved by means of the Archimedes assumption. The author wisely omits such proof, but he might have stated the fact, since the student easily gets an idea that this is an entirely new assumption independent of the rest.

A word may here be said about the nomenclature. Professor Halsted's innovations have not been generally accepted by text-book writers, but that proves nothing for or against his choice of terms. The word 'co-straight' will be objected to by purists as a hybrid; 'co-punctal' is better than 'concurrent,' if the idea of motion is to be eliminated; while the word 'sect' seems preferable to 'segment,' which is used to designate something else. The French translator has rendered 'sect' by 'segment,' 'co-straight' has for its equivalent 'collineaire,' and 'co-punctal' is translated 'coponctuelle.' We note with pleasure that the "Committee of Fifteen" has been kind enough to allow the use of the word "congruent." It may be that the author of *Rational Geometry* will live to see the day when a future more liberal "Committee of Fifteen" shall accept his nomenclature, if not *in toto*, yet the greater number of his terms.

As regards print, binding, and general appearance, the French edition is a real work of art. We shall say nothing about Professor Barbarin's translation; a competent judge, Professor Laisant, says: "Professor

* *Vereinfachungen des Hilbertschen System der Kongruenzaxiome*, von Arthur Rosenthal in Munchen.

† Assumption III, 5 is: "If an angle (h, k) is congruent as well to the angle (h', k') as also to the angle (h'', k'') then is also the angle (h', k') congruent to the angle (h'', k'').

Halsted's book is lucidly written, the translation is at once faithful and smooth."

We congratulate Professor Halsted on the well-merited success that his book has attained. Sooner or later it will no doubt exercise a profound influence on the teaching of geometry in our country.

A METHOD FOR THE SOLUTION OF SIMULTANEOUS QUADRATIC EQUATIONS OF THE SYMMETRIC TYPE.

By B. E. MITCHELL, Nashville, Tennessee.

The following method is suggested but not developed in Fine's *College Algebra.*

The most general quadratic of the symmetric type between two variables is $a(x^2+y^2)+2bxy+c(x+y)+d=0$.

Let x and y be the roots of $u^2+pu+q=0$. Then

$$u=\begin{cases} x=\frac{1}{2}[-p\pm\sqrt{(p^2-4q)}] \\ y=\frac{1}{2}[-p\mp\sqrt{(p^2-4q)}] \end{cases}\ldots(\mathrm{I}).$$

The double sign is used in each because they are the roots of a symmetric equation.

From the theory of quadratic equations we have

$$x+y=-p\ldots(1),$$

and $\quad xy=q\ldots(2),$

from which we calculate $\quad x^2+y^2=p^2-2q\ldots(3).$

These are the three symmetric forms that occur in the general equation above, and are the only ones of the second degree.

Let us solve two simultaneous equations by this method.

$$\begin{cases} a_1(x^2+y^2)+2b_1xy+c_1(x+y)+d_1=0\ldots(1). \\ a_2(x^2+y^2)+2b_2xy+c_2(x+y)+d_2=0\ldots(2). \end{cases}$$

Then

$$\begin{cases} a_1p^2-2(a_1-b_1)q-c_1p+d_1=0\ldots(3). \\ a_2p^2-2(a_2-b_2)q-c_2p+d_2=0\ldots(4). \end{cases}$$

Each of the equations (3) and (4) is linear in q. Solving them for q, we have

$$2q=\frac{a_1p^2-c_1p+d_1}{a_1-b_1}=\frac{a_2p^2-c_2p+d_2}{a_2-b_2}\ldots(5).$$

Considering equation (5) we have the following:

The coefficient of p^2 is $a_2b_1-a_1b_2=-\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ which we call A.

The coefficient of p is $a_1c_1-a_2c_2-b_1c_2+b_2c_1=-\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 0 \end{vmatrix}$ which we call B.

The constant term is $a_2d_1-b_2d_1-a_1d_2+b_1d_2=\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ d_1 & d_2 & 0 \end{vmatrix}$ which we call C.

Hence we have $Ap^2+Bp+C=0$ yielding generally two values for p, and from (5) we may calculate the two corresponding values of q. We have then the two equations in u:

$$u^2+p_1u+q_1=0,$$
$$u^2+p_2u+q_2=0,$$

which give four pairs of values for x and y when substituted in (I).

In the application of the method to numerical problems it will usually be found easier to use it as a process rather than the above equations as formulae.

Let us apply the method to three numerical problems taken from Fine's *College Algebra*.

(1) $\begin{cases} x+y=5. \\ xy+36=0. \end{cases}$

Hence $p=-5$ and $q=-36$, so $u^2-5u-36=0$, and $u=\begin{cases} x=9 \text{ and } -4. \\ y=-4 \text{ and } 9. \end{cases}$

(2) $\begin{cases} x^2+xy+y^2=21. \\ x+\sqrt{(xy)}+y=7. \end{cases}$ $\begin{cases} p^2-2q+q=21. \\ -p+\sqrt{q}=7. \end{cases}$ $\begin{cases} p^2-q=21. \\ p^2-q+14p=-49. \end{cases}$

$p=-5,\ q=4.\quad u^2-5u+4=0.\quad (u-4)(u-1)=0.$

$u=\begin{cases} x=1 \text{ and } 4. \\ y=4 \text{ and } 1. \end{cases}$

(3) $\begin{cases} x^2+3xy+y^2+2x+2y=8. \\ 2x^2+2y^2+3x+3y=14. \end{cases}$

Then $p^2-2p+q=8$, and $2p^2-3p-2q=14$.

Eliminating q by addition, $4p^2-7p-30=0$, $p=-2$ and $\frac{15}{4}$, then $q=-0$ and $\frac{23}{16}$. So $u^2-2u=0\ldots$(a), and $u^2+\frac{15}{4}u+\frac{23}{16}=0\ldots$(b).

From (a), $u=\begin{cases} x=0 \text{ and } 2. \\ y=2 \text{ and } 0. \end{cases}$

From (b), $u=\begin{cases} x=\frac{1}{2}\left[-\frac{15}{4}\pm\sqrt{\left(\frac{133}{16}\right)}\right]. \\ y=\frac{1}{8}\left[-15\mp\sqrt{(133)}\right]. \end{cases}$

This method may also be used where the equations are symmetric with respect to x and $-y$. In this case we have

$$x-y=-p,\ xy=-q,\ \text{and}\ x^2+y^2=p^2-2q.$$

By calculating other symmetric functions the method can be used successfully in solving many equations of higher degree than the second.

SOME CONSTRUCTIONS LEADING TO CONICS.

By F. H. HODGE, Franklin College, Indiana.

Among the courses that find a place in collegiate mathematics one is usually given which involves the treatment of plane curves. Certain loci are described in such a course and their equations are derived and the properties indicated by the equations are discussed. It seems desirable to have a stock of plane loci which can be simply described, and the derivation of whose equations does not present great difficulty, but which are not treated in current text books. Such loci could be assigned to pupils as original exercises, and they would serve to stimulate the capacity for independent thought.

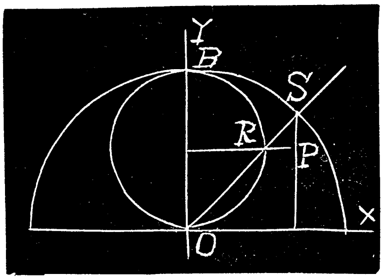

Fig. 1.

The four constructions which follow lead to conics. These are so simple and so directly analogous to a well known construction for the ellipse that it is scarcely possible that they are new though they are not found in current texts. The first two are described in detail, and the others merely suggested.

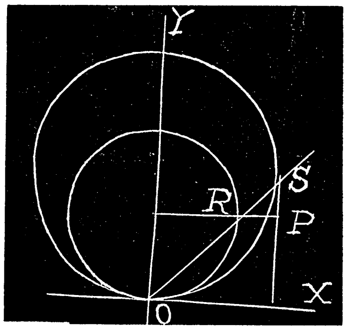

Fig. 2.

(1) Given two circles tangent internally at B and having the radius of the larger equal to the diameter of the smaller. Take the center of the larger circle as origin and the tangent to the smaller circle at that point as the x-axis, the y-axis being perpendicular to the x-axis. Draw a secant line through the origin, meeting the smaller circle at R and the larger circle at S. Through R draw a line parallel to the x-axis, and through S draw a line parallel to the y-axis. Call the point P in which these two lines meet. Required to find the locus of P as the secant line revolves about the origin as an axis. See Fig. 1.

(2) Here again are given two circles tangent internally, but no definite ratio is specified between the lengths of the radii. Take the point of tangency as the origin and the common tangent to the two circles as the x-axis, the y-axis being perpendicular to the x-axis. Draw the secant line as before and also the lines through R and S parallel to the x- and y-axes, respectively, and meeting at P. Required the locus of P as the secant line revolves. See Fig. 2.

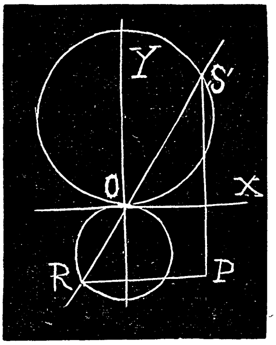

Fig. 3.

(3) This construction is similar to that of (2) except that the two circles are tangent externally, and the common tangent to the circles at the point of tangency is taken as the x-axis. See Fig. 3.

(4) Here the center of one circle lies on the circumference of the other circle. This center is taken as the origin, and the line joining it to the center of the other circle is taken as the y-axis. The secant and the other lines are drawn as in the other constructions, and the locus of P is required. See Fig. 4.

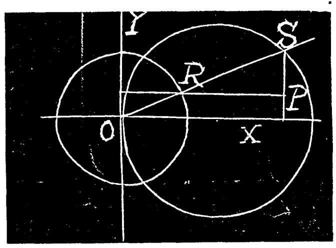

Fig. 4.

All four of these constructions can be shown by very elementary considerations to lead to conics, two of them giving ellipses, and the other two giving parabolas. Constructions similar to these can be made to yield an indefinite number of curves by simply replacing one or both of the circles by ellipses, parabolas, hyperbolas, or other curves. Such considerations, however, would lead us beyond the scope of an elementary course.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

368. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Michigan.

Solve the functional equation, $\dfrac{f(-x)}{f(x)}=r^{2x}$.

I. Solution by the PROPOSER.

Let $f(x)=A(x^2)+xB(x^2)$. Then the equation becomes

$$\frac{A(x^2)-xB(x^2)}{A(x^2)+xB(x^2)}=r^{2x}; \text{ whence } \frac{A(x^2)}{x\,B(x^2)}=\frac{1+r^{2x}}{1-r^{2x}}=-\frac{r^x+r^{-x}}{r^x-r^{-x}};$$

$$\text{or, } \frac{A(x^2)}{B(x^2)}=-\frac{r^x+r^{-x}}{(1/x)\ (r^x-r^{-x})}.$$

$\therefore A(x^2)=(r^x+r^{-x}).\phi(x^2)$, $B(x^2)=-\dfrac{1}{x}(r^x-r^{-x}).\phi(x^2)$, where $\phi(x^2)$ is any function of x^2.

$\therefore f(x)=\dfrac{2}{r^x}.\phi(x^2)$.

II. **Solution by S. LEFSCHETZ, Ph. D., The University of Nebraska.**

The given functional equation can be written:

$$f(-x)r^{-x}=f(x)r^x.$$

Hence $f(x)r^x=\phi(x)$, where ϕ is any even function of x.

$\therefore f(x)=\dfrac{\phi\ (x)}{r^x}$.

Ex. $\phi(x)=\sum\limits_0^\infty mA_m\cos mx$; $\phi(x)=\psi(x^2)$, etc.

Also solved by A. H. Holmes and J. Scheffer.

369. **Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.**

If $f(m)=(1+x)^m$, and $f(n)=(1+x)^n$, why not obviously $f(m).f(n)=(1+x)^{m+n}=f(m+n)$?

Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The proof given of $f(m)f(n)=f(m+n)$ is certainly incontestable; but it may also be proved directly thus: Differentiating with reference to m and n separately, we have

$$f(n)\,f'\,(m)=f'\,(m+n),\quad f'\,(n)\,f(m)=f'\,(m+n).$$

$\therefore f(n)\ f'\,(m)=f'\,(n)\ f(m)$; $\therefore \dfrac{f'\,(m)}{f(m)}=$a constant$=a$ (say).

$\therefore f(m)=a^m$.

Solutions of 367 were received from A. M. Harding, Elmer Schuyler, S. Lefschetz, and the Proposer.

GEOMETRY.

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394. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

The joins of the excentres to the corresponding vertices of the pedal triangle are concurrent.

I. Solution by WILLIAM HOOVER, Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let $ABCD$ be the triangle of reference; p_1, the perpendicular AP_1 upon the side A, analogous notation making $P_1P_2P_3$ the pedal triangle. Using trilinear coördinates, the center C_1 of the escribed circle touching a is given by $(-\alpha_1, \alpha_1, \alpha_1)$, and similarly for the other centers.

The point P_1 is given by $(0, p_1\cos C, p_1\cos B)$, and P_2, P_3 are set down by symmetry.

The equation to P_1C_1 is

$$\begin{vmatrix} \alpha & \beta & \gamma \\ -\alpha_1 & \alpha_1 & \alpha_1 \\ 0 & p_1\cos C & p_1\cos B \end{vmatrix} = \begin{vmatrix} \alpha & \alpha+\beta & \alpha+\gamma \\ -1 & 0 & 0 \\ 0 & p_1\cos C & p_1\cos B \end{vmatrix}$$
$$= \begin{vmatrix} \alpha+\beta & \alpha+\gamma \\ \cos C & \cos B \end{vmatrix} = \alpha(\cos B-\cos C)+\beta\cos B-\gamma\cos C=0\ldots(1).$$

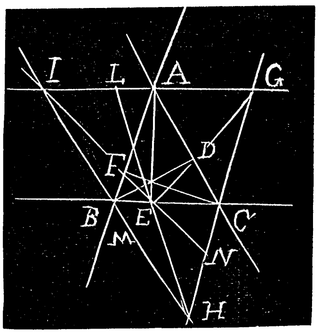
The equations to P_2C_2, P_3C_2 are, by symmetry,

$$-\alpha\cos A+\beta(\cos C-\cos A)+\gamma\cos C=0\ldots(2),$$
$$\alpha\cos A-\beta\cos B+\gamma(\cos A-\cos B)=0\ldots(3).$$

(1), (2), and (3) meet in a point, since

$$\begin{vmatrix} \cos B-\cos C & \cos B & -\cos C \\ -\cos A & \cos C-\cos A & \cos C \\ \cos A & -\cos B & \cos A-\cos B \end{vmatrix}$$
$$= \begin{vmatrix} \cos B & \cos B & -\cos C \\ -\cos C-\cos A & \cos C-\cos A & \cos C \\ \cos B & -\cos B & \cos A-\cos B \end{vmatrix} = 0.$$

II. Solution by N. L. RORAY.



Given the triangle ABC; D, E, and F vertices of the pedal triangle; G, H, and I centers of the ex-circles; GM, HL, and IN sects joining ex-centers with the corresponding vertices of the pedal triangle.

To prove: GM, HL, and IN concurrent.

Proof. $\dfrac{\triangle HIL}{\triangle BIA}=\dfrac{IL.IH}{IA.IB}$, $\dfrac{\triangle BIA}{\triangle HBE}=\dfrac{IB.BA}{BH.BE}$.

$$\therefore \frac{\triangle HIL}{\triangle HBE}=\frac{IL.IH.BA}{IA.BH.BE}=\frac{HI.HL}{HB.HE}. \quad \therefore \frac{IL}{IA}\cdot\frac{BA}{BE}=\frac{HL}{HE}.$$

Similarly, $\frac{GA}{GL}\cdot\frac{CE}{AC}=\frac{HE}{HL}$; $\frac{GN}{GC}\cdot\frac{AC}{AF}=\frac{IN}{IF}$; $\frac{HC}{HN}\cdot\frac{BF}{BC}=\frac{IF}{IN}$;

$$\frac{IB}{IM}\cdot\frac{AD}{AB}=\frac{GD}{GM}; \text{ and } \frac{HM}{HB}\cdot\frac{BC}{CD}=\frac{GM}{GD}.$$

$$\therefore \frac{IL.GN.HM}{GL.HN.IM}\cdot\frac{GA.IB.HC}{IA.BH.GC}\cdot\frac{CE.BF.AD}{BE.CD.AF}=1.$$

But by Ceva's Theorem, $\frac{CE.BF.AD}{BE.CD.AF}=1$.

Also, it is easily proved that $\frac{GA.IB.HC}{IA.BH.GC}=1$. $\therefore\frac{IL.GN.HM}{GL.HN.IM}=1$.

$\therefore GM$, HL, and IN are concurrent. Q. E. D.

397. Proposed by DAVID F. KELLEY, New York City.

If ABC be a semicircle and CD a perpendicular from C on the diameter AB, prove that the radius of the circle inscribed in the triangle ABC equals half the sum of the radius of the circle touching arc AC and the sides AD and DC of the triangle ADC, and the radius of the circle touching arc CB and sides DB and DC of triangle CDB, and that the centers of the three circles are collinear.

I. Solution by H. C. FEEMSTER, Professor of Mathematics, York College, York, Nebraska.

Let AB be the y-axis, CD the x-axis, r the radius of the given circle, $k=r-AD$.

Then $\begin{cases} x_1^2+(x_1+k)^2=(r-x_1)^2 \\ y_1=x_1, \end{cases}$ (1),

and $\begin{cases} x_{11}^2+(k-x_{11})^2=(r-x_{11})^2 \\ y_{11}=-x_{11}, \end{cases}$ (2),

are the relations that give the centers of the circles inscribed in ADC and BDC. The bisectors of the angles A and C intersect at

$$x=\frac{-2r+\sqrt{(2r^2-2rk)}+\sqrt{(2r^2+2rk)}}{2},$$

$$y=\frac{-2k+\sqrt{(2r^2+2rk)}-\sqrt{(2r^2-2rk)}}{2},$$

the center of the circle inscribed in ABC.

Solving the relations (1) and (2), we get

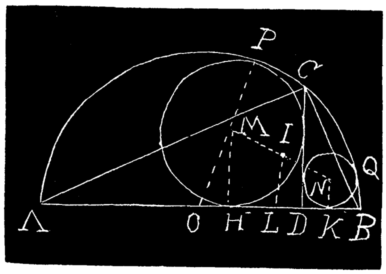
$$x_1 = y_1 = -k - r + \sqrt{(2r^2 + 2rk)},$$
$$x_{11} = -y_{11} = -r + k + \sqrt{(2r^2 - 2rk)}.$$

The x's represent the radii of the three circles, and satisfy the required conditions $x = \frac{x_1 + x_{11}}{2}$, and the centers are in a straight line since they satisfy the condition

$$\frac{y - y_1}{x - x_1} = \frac{y_{11} - y_1}{x_{11} - x_1}.$$

II. Solution by A. M. HARDING, A. M., University of Arkansas, and the PROPOSER.

Let M be the center of circle touching arc AC, sides AD and DC, and let H be the point where it touches AD. Let N be center of circle touching arc CB, sides CD and DB, and K the point where it touches DB. Also let I be center of circle inscribed in triangle ABC. $MH = HD$ since $\angle MDH = 45°$, and similarly, $NK = DK$. Therefore $HM + NK = HK$.



Again, by a well known theorem, $BC = BH$. Hence $\angle HCB = \angle CHB = \angle CAH + \angle ACH$, but $\angle BCD = \angle CAH$; therefore, $\angle HCD = \angle HCA$. Hence $\angle DAI + \angle ACH = 45°$. Again, $\angle ACI = 45°$, therefore $\angle DAI = \angle HCI$. Hence, the four points A, C, I, and H are concyclic. Hence, $\angle IHD = \angle ACI = 45°$. In a similar manner it can be shown that $\angle IKD = 45°$.

Now let L be the point where the circle inscribed in triangle ABC touches AB; then since $\angle ILH = 90°$, and $\angle IHL = \angle IKL = 45°$, line $IL = HL$ and $IL = LK$; but IL is the radius of circle inscribed in triangle ABC. Therefore $IL = \frac{1}{2}HK = \frac{1}{2}(MH + NK)$ by above.

Again, since L is the mid-point of HK, and IL, which $= \frac{1}{2}(MH + NK)$, is also parallel to MH and NK, it is easily seen that I is the mid-point of the line MN.

Also solved similarly by J. Scheffer.

CALCULUS.

318. Proposed by JOHN C. GREGG, Greencastle, Ind.

A thread is wound spirally n times around a cone, the radius of whose base is r, and slant height h, the turns being at uniform distance apart. If the thread is kept taut, what will be the length of the trace of its end on a horizontal plane?

I. Solution by B. F. FINKEL, Ph. D., Drury College.

A solution of this problem, by Henry Gunder, of Findlay (Ohio) College, was published in Vol. 9, page 199, of the *School Visitor*, published by John S. Royer. Professor Gunder's solution is wholly incorrect. So far as we know, no correct solution of the problem has ever been published. The problem as stated is indeterminate; for the reason that a string stretched taut on the surface of a cone cannot be in equilibrium unless held in position by the friction of the surface of the cone, by an adhesive substance, or else by pegs driven into the cone normal to the plane formed by the tangent to the string and the corresponding element of the cone. In the case of friction holding the string in position, it could not be wound on the cone nor unwound from it by merely taking hold of one end of the string and stretching it taut. Were a surface formed by conceiving a normal to the tangent of the string and the element of the cone, to move along the string, then the string could be wound on the cone by taking hold of one end of it and keeping it taut. The string would come in contact with this surface above the surface of the cone, and as the winding proceeds, the string would slip on this surface until it came in contact with the surface of the cone. The string would thus lie in the groove formed by the surface of the cone and the surface generated by a normal moving along the curve, formed by the proposed position of the string.

Since the unwinding as required in the problem is impossible, the problem is impossible of solution. If we suppose the string to adhere slightly to the cone, there are a number of ways which might be proposed to unwind the string. Of these, two would naturally suggest themselves. We might suppose, first, the string unwound in such a manner that the tangent to the projection of the string on the plane of the base and the unwound portion of the string are in the same plane; and second, that the unwound portion of the string and the altitude of the cone lie in the same plane. Both of these assumptions lead to very complicated computations. We shall now show, by a different method, what Professor Gunder obtained as a result, and also indicate a method of solution according to the two assumptions made above.

Let $C-AD'DBA$ be the cone, whose radius $OB=R$ and altitude $CO=h$; P, any point on the string, whose coördinates are (x, y, z); P', a consecutive point; $OP_1=\rho$, and $\angle BOP_1=\theta$. Draw the lines CPD, $CP'D'$, OD, OD', O_1P, and O_1K. P_1 is the projection of P; P_1' of P', and K_1 of K. P'_1P_1B is the projection of the string on the xy-plane; K_1P_1 is the projection of KP, and P'_1P_1 is the projection of IK and $P'P$.

The equation of the curve of the string is obtained by setting up a relation between $(z, \rho, \theta, n, R, h)$ or $(x, y, z, \theta, n, R, h)$.

In the similar triangles DP_1P and DOC, we have $R-\rho : R=z : h$, or

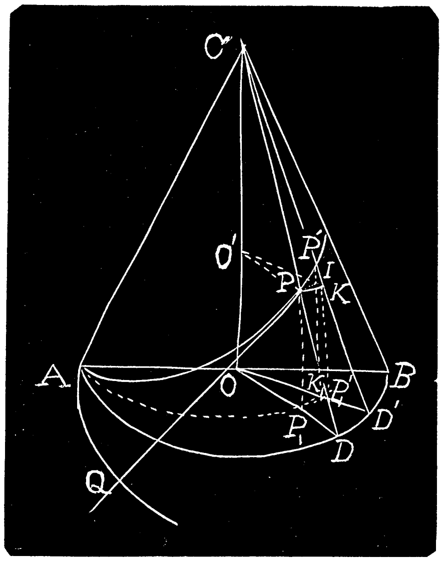
$$\rho=\frac{R(h-z)}{h}\dots(1).$$

Also, since the string makes n equidistant turns, we have arc $DB : 2\pi Rn = z:h$, or $R\theta : 2\pi nR = z:h$, or

$$z=\frac{\theta h}{2\pi n}\ldots(2);\quad \therefore \rho=\frac{R}{2\pi n}(2\pi n-\theta)\ldots(3).$$

the equation of the projection of the curve of the string on the xy-plane.

Also, $x=\rho\cos\theta=\frac{R}{h}(h-z)\cos\theta$ $=\frac{R}{2\pi n}(2\pi n-\theta)\cos\theta\ldots(4)$; $y=\rho\sin\theta$ $=\frac{R(h-z)}{h}\sin\theta=\frac{R}{2\pi n}(2\pi n-\theta)\sin\theta$ $\ldots(5)$, $z=\frac{\theta h}{2\pi n}$ are the parametric equations of the curve of the string. The equations of the tangent to the string are:



$$\frac{\alpha-x}{\frac{dx}{ds}}=\frac{\beta-y}{\frac{dy}{ds}}=\frac{\gamma-z}{\frac{dz}{ds}};\text{ or}$$

$$\frac{\alpha-x}{-\frac{R}{2\pi n}[\cos\theta+(2\pi n-\theta)\sin\theta]}$$

$$=\frac{\beta-y}{-\frac{R}{2\pi n}[\sin\theta-(2\pi n-\theta)\cos\theta]}=\frac{\gamma-z}{\frac{h}{2\pi n}}\ldots(6).$$

The coördinates of the point of intersection of the tangent with the xy-plane are found by letting $\gamma=0$ in (6). Hence,

$$\alpha=\frac{R}{2\pi n}[(2\pi n-\theta)(\cos\theta+\theta\sin\theta)+\theta\cos\theta]\ldots(7),\text{ and}$$

$$\beta=\frac{R}{2\pi n}[(2\pi n-\theta)(\sin\theta-\theta\cos\theta)+\theta\sin\theta]\ldots(8).$$

The length of the curve described by the point of intersection of the tangent with the xy-plane is

$$s=\int\sqrt{[d\alpha^2+d\beta^2]}=\frac{R}{2\pi n}\int_0^{2\pi n}\theta\sqrt{[4+(2\pi n-\theta)^2]}d\theta$$

$$=\frac{4R}{3\pi n}+\tfrac{2}{3}R\left(\pi n-\frac{2}{\pi n}\right)\sqrt{[1+\pi^2n^2]}+2R\log[\pi n+\sqrt{(1+\pi^2n^2)}].$$

This is the result obtained by Professor Gunder in the solution referred to above, and cannot be correct unless the string, in unwinding, is "taken up" at the same time; for the portion of the string unwound at any time is longer than the tangent to the string at the point of contact with the cone. The length of a portion of the string is

$$l=\int\sqrt{[dx^2+dy^2+dz^2]}=\frac{1}{2\pi n}\int_0^{\theta}\sqrt{[h^2+R^2(2\pi n-\theta)^2]}d\theta$$

$$=\frac{1}{2\pi n}\left[-\frac{R}{2}(2\pi n-\theta)\sqrt{\frac{h^2+R^2}{R^2}+(2\pi n-\theta)^2}-R\left(\frac{h^2+R^2}{R^2}\right)\log[(2\pi n-\theta)\right.$$

$$+\sqrt{\frac{h^2+R^2}{R^2}+(2\pi n-\theta)^2}]+\pi n R\sqrt{\frac{h^2+R^2}{R^2}+4\pi^2n^2}$$

$$\left.+R\left(\frac{h^2+R^2}{R^2}\right)\log(2\pi n+\sqrt{\frac{h^2+R^2}{R^2}+4\pi^2n^2}\right]$$

$$=\frac{1}{2\pi n}\left[\pi n\sqrt{[h^2+R^2+4\pi^2n^2R^2]}-\tfrac{1}{2}\sqrt{[2\pi n-\theta]}\sqrt{[h^2+R^2+(2\pi n-\theta)^2R^2]}\right.$$

$$\left.+R\left(\frac{h^2+R^2}{R^2}\right)\log\left(\frac{2\pi nR+\sqrt{[h^2+R^2+4\pi^2n^2R^2]}}{[2\pi n-\theta]R+\sqrt{[h^2+R^2+R^2(2\pi n-\theta]^2}}\right)\right].$$

The length of the corresponding tangent is

$$s=\sqrt{[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}=\frac{\theta}{2\pi n}\sqrt{[h^2+R^2(2\pi n-\theta)^2]}.$$

A comparison of these two expressions shows that the length of the unwound portion of the string is longer than the tangent to the point of contact. This fact also shows that the string cannot be unwound from the cone in the manner required by the problem unless it is held in contact with the cone by means of some force, as, for example, an adhesive substance.

Since the string must be held in position on the cone, we may choose any number of ways of unwinding it.

Of these ways, we have already mentioned two. First, suppose the string is unwound in such a way that the projection of the string on the xy-plane is the tangent to the curve $P_1'P_1B$. The equation of the trace of the string when unwound from the cone, by keeping it taut, is

$$OQ=[P_1Q^2+OP_1^2-2P_1Q.OP\cos\angle OP_1Q]^{\frac{1}{2}},$$

where $P_1Q=\sqrt{[PQ^2-P_1P^2]}=\sqrt{[s^2-z^2]}$, s being the unwound portion of the string, and $\angle QP_1O=\tan^{-1}[\rho\, d\theta/d\rho]$; or

$$OQ=\sqrt{s^2-z^2+\rho^2-\frac{2\rho\sqrt{[s^2-z^2]}}{[1+(2\pi n-\theta)^2]^{\frac{1}{2}}}}.$$

From this the length of the path may be found by approximate methods.

Second, suppose that the string is unwound so that the unwound part lies always in a plane with the axis of the cone. In this case the equation of the trace is $OQ=\sqrt{[s^2-z^2]}+\rho$, where s is the unwound portion of the string, and $\rho=OP_1$. The length of the trace in this case is

$$l=\int_0^{2\pi n}\{[(sds-zdz)(s^2-z^2)^{-\frac{1}{2}}+d\rho]^2+[\sqrt{(s^2-z^2)}+\rho]^2 d\theta^2\}^{\frac{1}{2}}.$$

A solution, by approximate quadrature, may be completed in a manner similar to that indicated in my solution of problem 309, pages 107-110, Vol. XIII of MONTHLY.

II. **Solution by PROFESSOR F. L. GRIFFIN, Reed College, Portland, Oregon.**

I. *Preliminary remark.* The problem is impossible in the absence of friction; since a stretched string can be in equilibrium on a "smooth" surface only along a geodesic line, and no geodesic line has the proposed location. Choosing the axis of the cone as the Z-axis, and the vertex as the origin, the equation of the surface is

$$f(x,\ y,\ z)\equiv x^2+y^2-z^2\tan^2\alpha=0,$$

where α is the half-angle of the cone, or $\arctan(r/h)$. The differential equations of a geodesic are

$$\frac{d^2x}{ds^2}=\nu\frac{\partial f}{\partial x}=2\nu x,\ \frac{d^2y}{ds^2}=\nu\frac{\partial f}{\partial y}=2\nu y,\ \frac{d^2z}{ds^2}=\nu\frac{\partial f}{dz}=-2\nu\, z\tan^2\alpha,$$

where s denotes the length of the arc and

$$\nu = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2} \Big/ \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}.$$

Combining the first pair of equations gives an integral, $x\frac{dy}{ds} - y\frac{dx}{ds} = c$, constant. But since for any point (x, y, z) on the cone, $\sqrt{[x^2+y^2]} = z\tan\alpha$, we have $x = z\tan\alpha\cos\phi$, $y = z\tan\alpha\sin\phi$, where ϕ denotes the polar angle of the projection of (x, y, z) on the horizontal reference plane. Then

$$\frac{dx}{ds} = \tan\alpha\left[\frac{dz}{ds}\cos\phi - z\sin\phi\frac{d\phi}{ds}\right], \quad \frac{dy}{ds} = \tan\alpha\left[\frac{dz}{ds}\sin\phi + z\cos\phi\frac{d\phi}{ds}\right],$$

whence the integral becomes $z^2\tan^2\alpha(d\phi/ds) = c$. (If $c=0$, then either $z=0$ permanently or else ϕ=constant, showing,— as is physically obvious,— that a thread cannot be thus wound up beginning at the vertex). Further,

$$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = \tan^2\alpha\left[\left(\frac{dz}{ds}\right)^2 + z^2\left(\frac{d\phi}{ds}\right)^2\right] + \left(\frac{dz}{ds}\right)^2$$

$$= \left[\left(\frac{dz}{d\phi}\right)^2\sec^2\alpha + z^2\tan^2\alpha\right]c^2/z^4\tan^4\alpha.$$

Separating variables, we have

$$\frac{dz}{z\sqrt{[z^2 - c^2\cot^2\alpha]}} = -\frac{\sin\alpha\tan\alpha}{c}\,d\,\phi,$$

whence, k being constant. $z = c\cot\alpha\sec(kc\cot\alpha - \phi\sin\alpha)$.

Evidently, for small enough values of $\sin\alpha$ the geodesic may wind several times around the cone before receding to infinity at a certain finite value of ϕ; but even these few turns are never at uniform distance apart. For, if adding 2π to ϕ merely adds some constant to z, it follows that $(dz/d\phi)$ is the same at $\phi+2\pi$ as at ϕ, since

$$\frac{z(\phi+\triangle\phi+2\pi) - z(\phi+2\pi)}{\triangle\phi} \equiv \frac{z(\phi+\triangle\phi) - z(\phi)}{\triangle\phi};$$

but clearly the period of $dz/d\phi$ is $2\pi/\sin\alpha$.

II. But if we waive this question, and suppose the surface of the cone

sufficiently rough to hold the thread in the proposed position; or if we think of the equivalent purely geometric problem, the required length, σ, of the trace of the end-point (ξ, η, ς), may be found, or at least expressed as a definite integral, as follows:

The requirement $z(\phi+2\pi)-z(\phi)\equiv m$, constant, does not define z as a function of ϕ, but is most simply satisfied by $z=h-m\phi/2\pi$, or $\frac{dz}{d\phi}=-\frac{m}{2\pi}$. This gives, retaining z for brevity:

$$\frac{dx}{d\phi}=\tan\alpha[-\frac{m}{2\pi}\cos\phi-z\sin\phi],\quad \frac{d^2x}{d\phi^2}=\tan\alpha[\frac{m}{\pi}\sin\phi-z\cos\phi],$$

$$\frac{dy}{d\phi}=\tan\alpha[-\frac{m}{2\pi}\sin\phi+z\cos\phi],\quad \frac{d^2y}{d\phi^2}=\tan\alpha\,[-\frac{m}{\pi}\cos\phi-z\sin\phi],$$

$$\left(\frac{ds}{d\phi}\right)^2=z^2\tan^2\alpha+\frac{m^2}{4\pi^2}\sec^2\alpha,\quad \frac{d^2s}{d\phi^2}=z\tan^2\alpha(-\frac{m}{2\pi})\,/\frac{ds}{d\phi}.$$

From $ds/d\phi$ it is easy to obtain s, the length of the curve on the cone to any point, or, in particular, to $\phi=2\pi n$, which gives the total length, l, of the string. Now we find from the equation of the tangent at any point (x, y, z) of this spiral curve:

$$\frac{\xi-x}{\frac{dx}{ds}}=\frac{\eta-y}{\frac{dy}{ds}}=\frac{\varsigma-z}{\frac{dz}{ds}}=l-s=\text{length yet unwound.}$$

Hence, $\varsigma=x+(l-s)\frac{dx}{d\phi}/\frac{ds}{d\phi}$, $\frac{d\xi}{d\phi}=(l-s)\dfrac{\frac{ds}{d\phi}\cdot\frac{d^2x}{d\phi^2}-\frac{dx}{d\phi}\cdot\frac{d^2s}{d\phi^2}}{\left(\frac{ds}{d\phi}\right)^2}$, and similarly for η, $d\eta/d\phi$. To get σ, the distance travelled by the projected point $(\xi, \eta, 0)$, we use $d\sigma^2=d\xi^2+d\eta^2$; thus

$$\left(\frac{d\sigma}{d\phi}\right)^2=(l-s)^2\,/\left(\frac{ds}{d\phi}\right)^4\tan^2\alpha\left[\left(\frac{ds}{d\phi}\right)^2\left(\frac{m^2}{\pi^2}+z^2\right)\right.$$

$$\left.+\frac{ds}{d\phi}\frac{d^2s}{d\phi^2}\left(\frac{m}{\pi}z\right)+\left(\frac{d^2s}{d\phi^2}\right)^2\left(\frac{m^2}{4\pi^2}+z^2\right)\right].$$

Every term in the right member being a known function of ϕ, we denote the entire member by $F(\phi)$, and write

$$\sigma=\int_0^{2n\pi}\sqrt{[F(\phi)]}\,d\phi.$$

Various reductions are possible in the integrand, but they do not seem to lead to an expression of the indefinite integral.

II. *Remark.* On a "rough" cone the thread may be wound following other curves than geodesics; and with sufficient friction the turns may be wound at uniform distance apart. But even this is impossible if the end of the thread be required to remain in some horizontal plane; so that the proposed problem is impossible (even with friction) if by "trace" we are to understand a curve actually travelled by the end of the thread. Let us prove this.

At every instant the unwound taut thread must lie in the plane tangent to the cone at the point of contact, P; so likewise must the line PT tangent to the curve on the cone at P. But the direction of PT (or of the next succeeding "element of arc") is determined by the intersection of the conical surface with the plane which is determined by the taut thread and the instantaneous direction of motion of its end-point. Since the tangent line PT lies both in the latter plane and in the tangent plane, PT must coincide with the taut thread which lies in these two planes. Thus in winding the thread about the cone, the unwound portion must at every instant be tangent to the curve on the cone.

Now if the end-point $(\xi,\ \eta,\ \varsigma)$ is to remain constantly in a plane at any distance g from the equation of PT, the tangent at $(x,\ y,\ z)$,

$$\frac{\xi-x}{\frac{dx}{ds}}=\frac{\eta-y}{\frac{dy}{ds}}=\frac{q-z}{\frac{dz}{ds}}=l-s, \text{ (the length yet unwound).}$$

Separating variables in the last equation, $dz/(q-z)=ds/(l-s)$, whence, p being constant, $p(q-z)=(l-s)$. From this, $p^2dz^2=ds^2=dx^2+dy^2+dz^2$, so that $(p^2-1)dz^2=dx^2+dy^2=\tan^2\alpha(dz^2+z^2d\phi^2)$, since as in (I) above, $r=\tan\alpha.z\cos\phi$, etc. Thus $dz^2(p^2-\sec^2\alpha)=z^2\tan^2\alpha.d\phi^2$, or

$$\frac{dz}{z}=\frac{\pm\tan\alpha}{\sqrt{[p^2-\sec^2\alpha]}}d\phi=\pm Kd\phi, \text{ say,}$$

the upper or lower sign being taken according as the winding proceeds away from or toward the vertex, and the constant K being evidently real for a real curve. The integral $z=A.e\pm K\phi$ shows both that the winding process cannot begin at the vertex, since z vanishes only if $A=0$ in which case $z\equiv 0$, and also that the turns are not at a uniform distance apart, since the addition of 2π to ϕ serves merely to multiply z by a constant factor.

III. Suppose now that the end-point (ξ, η, ς) moves so that the turns on the cone ("rough," of course) are equidistant as proposed, and let us find the length of the curve traced by the projection (ξ, η, 0).

The requirement of equidistant turns, viz., $z(\phi+2\pi)=z(\phi)+m$, where the constant m is negative if the turns approach the vertex, does not define z as a function of ϕ, but it can be most simply satisfied by taking $z=A+m\phi/2\pi$, A being any constant not zero. Then

$$dx=\tan\alpha\left[\frac{m}{2\pi}\cos\phi-z\sin\phi\right]d\phi,\quad dy=\tan\alpha\left[\frac{m}{2\pi}\sin\phi+z\cos\phi\right]d\phi,$$

where z is written for brevity. Then

$$ds^2=\left[\frac{m^2}{4\pi^2}\sec^2\alpha+\tan^2\alpha\left(A+\frac{m\phi}{2\pi}\right)^2\right]d\phi^2,$$

from which s can be obtained by a simple integration; and in particular the entire length l may be found in terms of n, α and the practically arbitrary A, m. Then to find the length, σ, of the path of (ξ, η, 0) we have

$$\xi=x+(l-s)\,\frac{dx}{d\phi}\Big/\frac{ds}{d\phi},\qquad \eta=y+(l-s)\frac{dy}{d\phi}\Big/\frac{ds}{d\phi};$$

$$\frac{d\xi}{d\phi}=(l-s)\,.\,\frac{\dfrac{ds}{d\phi}\dfrac{d^2x}{d\phi^2}-\dfrac{d^2s}{d\phi^2}\dfrac{dx}{d\phi}}{\left(\dfrac{ds}{d\phi}\right)^2},\quad \frac{d\eta}{d\phi}=(l-s)\,\frac{\dfrac{ds}{d\phi}\cdot\dfrac{d^2y}{d\phi^2}-\dfrac{d^2s}{d\phi^2}\dfrac{dy}{d\phi}}{\left(\dfrac{ds}{d\phi}\right)^2};$$

$$\left(\frac{d\sigma}{d\phi}\right)^2=\left(\frac{d\xi}{d\phi}\right)^2+\left(\frac{d\eta}{d\phi}\right)^2=(l-s)^2\,(\ \ldots$$

as on bottom of page 107, except that the former m is here $-m$, etc., to former conclusion).

IV. A thread whose turns about a cone have been made uniformly distant by the help of friction may be *unwound* without the end-point retracing its former path; and as the unwound portion need not now be tangent to the curve on the cone, the end-point may travel in a horizontal plane. Let us find the length of its path, again taking arbitrarily $z=A+m\phi/2\pi$. The taut thread must still be in the tangent plane: $(\xi-x)x+(\eta-y)y+(q-z)(-z\tan^2\alpha)=0$; and $(\xi-x)^2+(\eta-y)^2+(\varsigma-z)^2=(l-s)^2$. Now let $\xi-x=(l-s)\cos a$, $\eta-y=(l-s)\cos b$, $q-z=(l-s)\cos c$, introducing the direction-cosines of the

string. Then $\cos^2 a+\cos^2 b=\sin^2 c$, and $\cos a.\cos\phi+\cos b\sin\phi=\cos c\tan\alpha$, (dividing x and y by $z\tan\alpha$). Combining the first of these equations with that obtained by squaring the second, we find $\cos a\sin\phi-\cos b\cos\phi=\pm\sqrt{[1-\cos^2 c\sec^2\alpha]}$. Thus

$$\cos a=(\xi-x)/(l-s)=\pm\sin\phi\sqrt{[1-\cos^2 c\sec^2\alpha]}+\cos c.\tan\alpha.\cos\phi;$$
$$\cos b=(\eta-y)/(l-s)=\mp\cos\phi\sqrt{[1-\cos^2 c\sec^2\alpha]}+\cos c.\tan\alpha.\sin\phi.$$

Since x, y, z, s, $\cos c$ are all known functions of ϕ, these equations furnish $d\xi$ and $d\eta$ in terms of ϕ and $d\phi$, and thus the problem of finding σ is reduced to a quadrature.

NOTES AND NEWS.

Mr. C. H. Forsyth of Michigan University has accepted the chair of mathematics in Eureka College, Eureka, Illinois. M.

Mr. C. A. Barnhart, assistant in the University of Illinois, has accepted the chair of Mathematics in Carthage College, Carthage, Illinois. M.

Dr. R. K. Morley, instructor in the University of Illinois, has accepted an assistant professorship of mathematics in Worcester Polytechnic Institute, Worcester, Mass. M.

Dr. G. F. McEwen, instructor in mathematics in the University of Illinois, has accepted a position in the Marine Biological station of the University of California. This station is located at Lajolla, Cal. M.

A circular, bearing the signature of about seventy prominent mathematicians, calls attention to the fact that during August of the present year Professor Felix Klein of Göttingen, Germany, will reach the fortieth anniversary of his appointment as professor of mathematics in the University of Erlangen. It is proposed to give him some token of the thanks and the good wishes of his fellow mathematicians in view of his great services to mathematical progress, and of his excellent work as a mathematical teacher. Attention is called especially to the great influence of the so-called *Erlanger Programm,* which was translated into English by Professor Haskell, and which was published in the *Bulletin of the New York Mathematical Society,* volume 2 (1893), page 215, under the title, "A Comparative Review of Recent Researches in Geometry." It may be added that Klein is the president of the International Commission on the Teaching of Mathematics, which is expected to report to the International Congress of Mathematicians at its fifth meeting, which is to be held at Cambridge, England, during the coming August. M.

During the Summer Session of the University of Illinois, June 17 to August 9, 1912, the following mathematical courses will be offered: Advanced algebra, plane trigonometry, analytic geometry, differential calculus, integral calculus, differential equations, and projective geometry. Graduate credit may be granted only for work in the last two of these courses. M.

It may interest mathematicians to learn of a valuable series of card-catologue cards which the Library of Congress will have ready for distribution next July. This series is the beginning of a Dictionary Catalogue of all articles in the *Encyklopädie der Mathematischen Wissenschaften* and *Encyclopēdie des Sciences Mathēmatiques*. Brown University has already supplied copy for the catalogue of all parts of these works which have been published. It will continue to furnish copy for further cards as the various parts of the encyclopædias appear. R. C. ARCHIBALD.

The latest number of the *Revue Semestrielle*, covering the period of six months from April to October, 1911, classifies the mathematical literature which appeared during this period under about 250 different headings, excluding the sub-headings represented by small letters of the Roman and the Greek alphabets. The largest number of references appear under the headings of biography and various considerations on the philosophy and the teaching of mathematics. The other headings under each of which there are more than twenty references to literature appearing during the given six months are as follows: Functions of real variables, series and infinite developments, electrodynamics, thermodynamics, light, surfaces in general and lines traced on a surface, theory of equations, functional equations, calculus of probability, systems and families of surfaces, elasticity, determinants, integral calculus, algebraic and circular functions, particular linear differential equations, indeterminate analysis of order higher than the first, plane and spherical curves, dynamics of solids and of material systems, rational hydrodynamics, and the history of mathematics in the twentieth century. The numbers of these references are useful to determine the fields of greatest present mathematical activity. Other facts must, however, be also considered. M.

BOOKS.

Non-Euclidian Geometry. A Critical and Historical Study of Its Development. By Roberto Bonola, Professor in the University of Pavia. Authorized English translation with additional appendices, by H. S. Carslaw, Professor in the University of Sydney, N. S. W., with an introduction by Federigo Enriques, Professor in the University of Bologna. 8vo. Red cloth. xii+263 pages. Price, $2.00. Chicago: The Open Court Publishing Co.

This is, as it purports to be, a critical and historical study of non-Euclidean geome-

try. The first chapter deals with attempts to prove Euclid's parallel postulate, beginning with the Greeks, Euclid, Proclus, etc., and ends with the investigations of Wallis. The second chapter treats of the forerunners of non-Euclidean geometry, beginning with Gerolamo Sacchiri and ends with Wachter and Thibaut. Thibaut is alleged to be responsible for the erroneous proof of the angle-sum, by starting at one corner of a triangle and running the side along in its own trace until the next vertex is reached, then turning the side through an angle until it coincides with next adjacent side, and so on. Chapters III and IV deal with the founders of non-Euclidean geometry, beginning with Gauss and Bolyai and ending with Battaglini and Beltrami. Chapter V. has to do with the later developments of non-Euclidean geometry. In the five appendices are treated in order, The fundamental principles of statics and Euclid's postulate, Clifford's parallels and surface and sketch of Clifford-Klein' problem; the non-Euclidean parallel construction and other allied constructions; the independence of projective geometry from Euclid's postulate; and the impossibility of proving Euclid's parallel postulate. The book gives a most satisfactory treatment of the subject and will be welcomed by all interested teachers of Geometry. The publishers have done their part well in presenting this translation to English speaking mathematicians. F.

An Elementary Treatise on Statics. By S. A. Loney, M. A., Professor of Mathematics at the Royal Holloway College (University of London). Sometime Fellow of Sydney Sussex College, Cambridge. Large 8vo. Red Cloth. viii+393 pages. Price, $4.00 net. Cambridge, Eng.: The University Press. G. P. Putnam's Sons, American Agents.

This book is intended to serve as a companion to the author's excellent treatise on *Dynamics of a Particle and of Rigid Bodies.* It covers most of the usual subjects of Statics and requires on the part of the student a knowledge of the calculus and of solid geometry. A large collection of interesting problems are placed at the end of each chapter and after many of the articles. The treatment of the various subjects is in keeping with all the other works which Professor Loney has written. There is one thing, however, which most, if not all of his works lack, and that is a good index. The typography and binding is all that could be desired. F.

Kimball's Commercial Arithmetic. Prepared for use in Normal, Commercial, and High Schools and for the Higher Grades of the Common Schools. By Gustavus S. Kimball, Author of *Business English, Word Book, Business Speller,* etc.. 8vo. Cloth and Leather back. viii+418 pages. New York: G. P. Putnam's Sons.

A very good book for the commercial student. All the ordinary business transactions, such as Banking, Stocks and Bonds, and Insurance, are quite fully treated.

As is the usual way in books of this kind, many useful short-cuts for rapid calculation are clearly explained and illustrated by numerous illustrative examples. Of course, short-cuts are of little value educationally, but are useful to the accountant. The book is well conceived and well arranged. The answers are bound in a separate volume. F.

The Modern Locomotive. By C. Edgar Allen, A. M. I. Mach. E.; A. M. I. E. E. Small 8vo. Cloth, ix+174 pages. Cambridge, Eng.: The University Press. G. P. Putnam's Sons, American Agents.

This little volume is one of the Cambridge Manuals of Science and Literature. Its object is to sketch the general "principles governing the design and working of a modern locomotive and to trace the broad lines of development from its comparatively simple predecessor of twenty-five or thirty years ago." Much attention is given to combustion, transfer of heat, steam production, super heating, compounding, feedwater heating, resistance and stability. It has many valuable suggestions for the engineer. F.

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SOME APPRECIATIVE REMARKS ON THE THEORY OF NUMBERS.

By DR. G. A. MILLER, University of Illinois.

In volume 10, 1903, of this journal the writer of the present note gave a brief list of appreciative remarks on the theory of groups with a view to furnishing an easy means to gain a knowledge of some important elements of this theory. The present brief list of appreciative remarks on the theory of numbers is intended to serve a similar purpose. It should be remembered that such extracts frequently require some modifications, but they direct attention to central truths of great value.

In the introduction to Reid's *Elements of the Theory of Algebraic Numbers*, 1910, Professor Hilbert asserts that "up to the present there is indeed no other science so highly praised by its devotees as is the theory of numbers." Some American students of mathematics might at first be inclined to attach little significance to these words in view of the fact that the number of devotees of this subject may be supposed to be very small. It should be observed that the number of those who study the theory of numbers, especially in Germany, is not insignificant. If it is remembered that Kummer's classes in this subject at the time of his greatest popularity in the University of Berlin, numbered at least 250,* and that classes of more than 100 are not uncommon now, in this leading German university, it is evident that conclusions as regards the number of devotees of this subject should not be based on conditions in our own universities.

In 1907 Minkowski began the preface of his book, entitled *Diophantische Approximation*, with the following words: "Integral numbers are the source (Urquell) of all mathematics. By this I understand not only the old view according to which the concept of continuity can also be deduced out of the consideration of discrete aggregates. I think much more about later results in using these words. The facts that the theory of the division of the circle dominates the theory of expotential functions and that the el-

* *Festschrift zur Feier dês 100 Geburtstages*, Edward Kummers, 1910, p. 15.

liptic functions may be comprehended by means of modular equations, inspire the confident belief that deepest relations in analysis are of an arithmetical nature." It may be of interest to note in this connection that the 15th reprint, 1910, of the Prospekt of the great German mathematical encyclopedia announces that part 5 of the Nachtraege to Band II will be devoted to relations between function theory and number theory, "Beziehungen zwischen Funktionentheorie und Zahlentheorie."

More than sixty years ago the noted French mathematician, Poinsot, expressed himself, in the *Journal de Mathêmatiques*, volume 10, page 2, as follows: It seems that the authors have regarded, since a long time, the theory of numbers as a queer speculation which is connected with nothing either in analysis or in geometry, and consequently offers to the intellect only truths which are more curious than useful. One finds scarcely any trace of it in the ordinary treatises on arithmetic and algebra. However, it is easy to see, after little reflection, that this transcendent arithmetic is the origin and source of real algebra. This is a truth which can be established by reason as I shall show directly, but which may also be proved, in a manner, by experience. For we believe that the little which is added from time to time to algebra comes from the little which is discovered at intervals in the science of the properties of numbers. An especial example of this kind is furnished by the felicitous relations connecting the algebraic solution of binomial equations of all degrees and the nature of prime numbers, according to which the circle can be divided into equal parts by means of the rule and the circle. This was an unexpected and very remarkable step which the theory of numbers furnished at the same time to algebra and to geometry.

"I believe that we shall at some time succeed to arithmetize all mathematical disciplines with the exception of geometry and mechanics; that is, to base them singly and solely on the concept of numbers in the most restricted sense, and hence cut off the modifications and extentions (namely the irrational and continuous quantities) of this concept, which were mainly due to applications to geometry and mechanics."* God made the integers; all the rest is the work of men, "Die ganzen Zahlen hat der lieve Gott gemacht, alles andere ist menschenwerk."† "It is a remarkable and suggestive fact that scarcely two hundred years after the discovery of the calculus, the higher mathematics has already exhibited a strong tendency to converge towards the oldest of all mathematical sciences, that of harmonious discontinuity — the theory of numbers."‡

While it is natural to suppose that specialists in number theory have made the most favorable remarks in regard to this subject it is not true that a high opinion of this subject is confined to such specialists. Among extreme specialists in synthetic geometry, Jacob Steiner occupies a promi-

* Kronecker, *Crelle Journal*, vol. 101 (1887), p. 338.

† Kronecker, Weber, *Mathematische Annalen*, vol. 43 (1893), p 15.

‡ Cole, *American Journal of Mathematics*, vol. 9 (1887), p. 46.

nent place; yet, according to an account by Lampe*, Jacob Steiner advised his students especially to study the theory of numbers on the ground that this subject was eminently suited to cultivate acuteness. Steiner is said to have reprimanded severely a young student who wanted to devote himself exclusively to the study of systhetic geometry as Steiner had done himself. It is said that Steiner remarked that not all those who would say to him, Lord, Lord, could enter the Kingdom of Heaven.

The scientific developments of the fundamental facts of the theory of numbers are largely due to Gauss and they were first published in the classic Disquisitiones Arithmeticae, 1801. In a biographical sketch of Gauss entitled *Gauss zum Gedaechtniss*, 1856, Waltershausen stated that Gauss called mathematics the queen of the sciences and arithmetic the queen of mathematics. Although this science descends often to render service to astronomy and other natural sciences, yet it deserves under all conditions the first rank.

"Goepel was first attracted by the higher theory of numbers, like many of those who are chosen for mathematical speculations."† "There is a theory which has been equally useful to me in all my researches, namely, that of the groups formed by the linear substitutions. In fact, these substitutions play a preponderant role in the study of linear equations and in that of arithmetic forms. It is to this circumstance that one ought to attribute the interrelations, often unexpected, which I shall note later between the theory of numbers and the theory of Fuchsian functions,— theories which, moreover, do not at first appear to have any point of contact."‡ "From the problems which we have just examined we see that the three fundamental branches of mathematics, namely, the theory of numbers, algebra, and the theory of functions, are most intimately related; and I am convinced that the theory of analytic functions of several variables will make decided progress if one shall arrive at the discovery and the study of functions which, in the domain of any given algebraic numbers, play a role analogous to that played by the expotential functions in the domain of the rational numbers, and by the elliptic modular functions in the domain of quadratic imaginary numbers."§

It would be easy to extend this list of quotations very much but the object of the present note is attained if a sufficient number of quotations have been given to exhibit the importance and the bearing of the theory of numbers. Just as the engineer is inclined to hasten into his profession without a sufficient training in mathematics, so the student of mathematics is often tempted to proceed to fields for which he has been imperfectly prepared. While many mathematical subjects are mutually helpful, the question as to which should be regarded as the more fundamental is sometimes of great importance.

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* Lampe, *Bibliotheca Mathematica*, 3rd Series, vol. 1 (1900), p. 134.
† C. G. J. Jacobi, *Crelle Journal*, vol. 35 (1847), p. 313.
‡ Poincaré, *Note sur les travaux scientifiques* (1884), p. 7.
§ Hilbert, *Paris Intenational Mathematical Congress* (1900), p. 90.

A GEOMETRIC EXAMPLE OF AN INDETERMINATE FORM.

By ARTHUR C. LUNN, University of Chicago.

In the prolate ellipsoid of revolution generated by turning the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ around the x-axis be cut by the plane $x=c$ through one focus, the area of the smaller part, between this focal plane and the nearer vertex, is given by the integral

$$S=\frac{2\pi b}{a^2}\int_c^a \sqrt{[a^4-(a^2-b^2)x^2]}\,dx.$$

When the constants are expressed in terms of the eccentricity e and the distance q between the focus and vertex, by means of the relations

$$c=ae,\quad b=a\sqrt{[1-e^2]},\quad q=a(1-e),$$

the integral reduces to

$$(1)\qquad S=\pi q^2\sqrt{\frac{1+e}{(1-e)^3}}\{\sqrt{[1-e^2]}-e\sqrt{[1-e^4]}+\frac{\sin^{-1}e-\sin^{-1}e^2}{e}\}.$$

If now the distance q be kept constant, the limiting surface (for $e=1$) is the paraboloid of revolution generated by $y^2=4qx$ (with a convenient change of axes); and the corresponding area is

$$S_1=\frac{\pi}{q}\int_0^{2q} y\sqrt{[y^2+4q^2]}\,dy$$

which reduces to

$$(2)\qquad S_1=\frac{8\pi q^2}{3}(\sqrt{8}-1).$$

It is thus apparent that as e approaches unity the limit S should be S_1. But the expression for S is then formally indeterminate, and verification is not directly practicable by the customary method of successive differentiation with respect to e, the reason being the presence of the factor $(1-e)$ raised to a fractional power, which cannot be reduced to a constant by repeated differentiation. It may be noticed also that the point $e=1$ is a branch-point for $\sin^{-1}e$ and for $\sin^{-1}e^2$, where two values of a many-valued function come together.

This suggests the change of variable $1-e=t^2$; then S as a function of t may be treated by successive differentiation; or, what is at bottom the same thing, let the various terms in the brace be expanded in integral powers of t, with the values at $t=0$ suitably chosen thus:

$$\sqrt{[1-e^2]}=\sqrt{[2]}\,t-\tfrac{1}{4}\sqrt{[2]}\,t^3\ldots.$$
$$e\sqrt{[1-e^4]}=2t-\tfrac{7}{2}t^3\ldots.$$
$$\sin^{-1}e=\frac{\pi}{2}-\sqrt{[2]}\,t-\frac{\sqrt{2}}{12}t^3\ldots.$$
$$\sin^{-1}e^2=\frac{\pi}{2}-2t+\tfrac{1}{6}t^3\ldots.$$

then in the expression for S, the factors t^3, to which the indeterminateness is due, may be cancelled, and the leading coefficients are then such as to show that the limit of S is S_1 as t approaches zero.

A contrasting case occurs with respect to the entire area of the ellipsoid, which is

$$2\pi a^2\sqrt{[1-e^2]}\left\{\frac{\sin^{-1}e}{e}+\sqrt{[1-e^2]}\right\},$$

the limit of which, as e approaches zero, is the area of a sphere of radius a. Here the indeterminateness occurring in $\frac{\sin^{-1}e}{e}$ is of integral order.

Many similar examples could be cited, for instance, that of the center of gravity or center of pressure of part of an ellipse and the corresponding part of the limiting parabola, where the limit of an indeterminate form, discovered by geometric or physical considerations, is to be analytically verified.

CERTAIN INTEGRATION FORMULAE USEFUL IN NUMERICAL COMPUTATION.

By S. A. COREY, Hiteman, Iowa.

In developing a function of a variable by Taylor's, Maclaurin's, or the Euler-Maclaurin formulae, the evaluation of the successive derivatives is a fundamental and in most cases indispensable operation. In many instances the higher derivatives become exceedingly complex and their evaluation correspondingly difficult. In such cases frequently the labor involved in finding results of the required degree of accuracy by these formulae becomes prohibitively great, and other methods are resorted to, involving usually some of the formulae of the Calculus of Finite Differences. The formulae given below, some of which are probably new and others old except as to the remainder term, will, it is believed, prove useful in many such cases, for the reason that the use of the higher derivatives is not involved except in the remainder term, in which term a roughly approximate value usually suffices. In certain cases it will be found desirable to develop the given function to a limited number of terms by Taylor's or Maclaurin's formula, to differentiate this series the number of times required to get an expression for the derivative involved in the remainder term, and then from this expression to determine roughly the value of the remainder term. Another method of indicating the degree of accuracy assured in cases where the value of the remainder term is not easily determined is to develop the function by two of the formulae given below, and then to compare the results obtained. It is usually safe to assume that the difference in the results obtained may be substituted for the remainder term in either of the two formulae employed. It will be noted that the remainder term, as given below, is expressed in a form similar to that frequently employed in Taylor's formula, and that it is therefore always possible to compare the accuracy assured with the accuracy assured by the use of Taylor's formula.

From the manner of deriving these formulae it will appear that it is not practicable to give all the formulae of like form, obtainable from the general formula (c), but the list here given is, probably, sufficiently complete to satisfy all ordinary needs. Should others be desired they may, of course, be obtained from the general formula. In fact, were it not for the considerable amount of labor involved in deriving these formulae one at a time, all these formulae could be thus obtained and there would be no occasion for appending the following rather formidable list.

These formulae are all based on the so-called Euler-Maclaurin formula, a semi-convergent formula which has been recently studied by E. Borel, E. W. Barnes,* and other European mathematicians, but to which but scanty

* Borel, *Lecons sur les sèries divergentes*, Chap. III.
Borel, *Acta Mathematica*, Vol. XX, page 360.
Barnes, *Quarterly Journal of Mathematics*, Vol. XXXV, pages 175-188.
Barnes, *Proc. London Mathematical Society*, Series 2, Vol. 3, page 253.
Cantor, *Geschichte der Mathematik*, Vol. 3, pages 635, 661.

reference is made in our American mathematical literature. It would seem a greater use of this formula in our American text books would be advisable.

William Fleetfoot Sheppard, in his article on "Mensuration" in the 11th edition of the Encyclopedia Brittannica, volume XVIII, pages 142-3-4, paragraphs 68 to 79 inclusive, gives a very satisfactory reference to this class of formulae, to which the reader is referred. In paragraph 71 he gives a general method of constructing some of these formulae, but it should be carefully noted that the subscripts of C which he uses are determined by a system of notation somewhat different from that herein used. In paragraph 79 he gives another general form involving an expression for the remainder, but the use of the formulae in general numerical work is not emphasized, and the reader is likely to infer that their use is confined chiefly to the subject under discussion.

The Euler-Maclaurin formula may be written,

$$f(a+x)=f(a)+\frac{x}{2m}\{f'(a)+f'(a+x)+2[f'(a+\frac{x}{m})+f'(a+\frac{2x}{m})$$

$$+f'(a+\frac{3x}{m})\ldots.+f'(a+\frac{m-1}{m}x)]\}-\frac{B_1 x^2}{m^2.2!}[f''(a+x)-f''(a)]$$

$$+\frac{B_2 x^4}{m^4.4!}[f^{\mathrm{iv}}(a+x)-f^{\mathrm{iv}}(a)]\ldots.$$

$$+(-1)^n\frac{B_n x^{2n}}{m^{2n}.(2n)!}[f^{2n}(a+x)-f^{2n}(a)]+S\ldots.(\mathrm{a}).$$

B_n being Bernoulli's nth number.

Expressed briefly and for clearness in what follows, this becomes:

$$f(a+x)=F(x,m)+\frac{k_2}{m^2}+\frac{k_4}{m^4}+\frac{k_6}{m^6}+\frac{k_8}{m^8}\ldots.+\frac{k_{2n}}{m^{2n}}+S\ldots.(\mathrm{b}),$$

in which k_2, k_4, k_6, ..., k_{2n}, are independent of m, and S is an expression for the remainder after the general term.

If $(r+1)$ values of m be taken, r of the quantities k, say k_{2i}, $k_{2(i+1)}$, $k_{2(i+2)}$, ..., $k_{2(i+r-1)}$, may be eliminated between the $(r+1)$ resulting equations, and the result expressed in determinant form as follows:

$$f(a+x)=\frac{\begin{vmatrix} M_1+S_1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \ldots & m_1^{-2(i+r-1)} \\ M_2+S_2 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \ldots & m_2^{-2(i+r-1)} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & & \cdot\ \cdot \\ M_{(r+1)}+S_{(r+1)} & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \ldots & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}}{\begin{vmatrix} 1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \ldots & m_1^{-2(i+r-1)} \\ 1 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \ldots & m_2^{-2(i+r-1} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & & \cdot\ \cdot \\ 1 & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \ldots & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}} \ldots. (c),$$

where M_1, M_2, ..., $M_{(r+1)}$, represent the approximate values of $f(a+x)$, obtained by (b) for m_1, m_2, ..., $m_{(r+1)}$, respectively, by employing all the terms preceding the term $\frac{k_{2i}}{m^{2i}}$, and where S_1, S_2, ..., $S_{(r+1)}$, represent the corresponding remainder terms, *i. e.* the remainder after the term $\frac{k_{2(i+r-1)}}{m^{2(i+r-1)}}$ in each of the $(r+1)$ equations.

Splitting up the determinant of the numerator into two other similar determinants, one containing the M elements and the other the S elements, we have, by letting D represent the determinant,

$$f(a+x)=D_M/D+D_S/D\ldots. (d).$$

To rid the indices of the elements of these determinants of the minus sign, divide each element in a row by the last element in that row. Now reduce the order by taking the co-factors of all the elements in the first column. These co-factors have evidently as factors $(m^2_u-m^2_v)$, u and v taking any value assumed by the subscripts of m, except that of the element, the co-factor of which is involved, and except $u=v$. Their evaluation follows readily.

The remainders, S, used in determining the remainders in the following formulae, were obtained from the solution of problem 237, Calculus, given in the June-July, 1907, number of THE AMERICAN MATHEMATICAL MONTHLY, to which reference must be had for an understanding of the method used in their derivation. Each remainder term in the following is dependent for its value on but *one* higher derivative, and for that reason the limits of its value are fixed, while the form given by Mr. Sheppard is a function of *all* derivatives higher than a certain derivative, and usually becomes *divergent* if a sufficiently large number of terms be taken.

Instead of retaining in the following formulae the M_1, M_2, ..., $M_{(r+1)}$, used in (c), it is convenient to employ a slightly different notation, viz., C_m represents the chordal area of the curve $y=f'(a+x)$, where the

value of y is taken at each of the $(m+1)$ equidistant intervals, 0, x/m, $2x/m$, $3x/m$, ..., $(m-1)x/m$ and x, thus

$$C_m = \frac{x}{m}\left[\frac{f'(a)+f'(a+x)}{2} + f'\left(a+\frac{x}{m}\right) + f'\left(a+\frac{2x}{m}\right) + \ldots + f'\left(a+\frac{(m-1)}{m}x\right)\right].$$

In the formulae involving y, the first extreme ordinate is invariably y_o, and the last extreme ordinate y_n, where n has the highest value of the subscript given in that formula. The distances of the intermediate ordinates y_r (where $o<r<n$) and the last ordinate y_n from the first extreme ordinate are *proportional to their subscripts.* The ordinates are therefore not *all* equidistant *in cases where the smaller values of m are not all factors of the largest value.*

Where C is given more than one subscript, thus $C_{m,t}$, it is understood that to the expression for the chordal area C_m has been added the sum of the first t terms of the series $\frac{k_2}{m^2}+\frac{k_4}{m^4}+\frac{k_6}{m^6}+$ etc., in **(b)**, where

$$k_{2n} = +(-1)^n \frac{B_n x^{2n}}{(2n)!}[f^{2n}(a+x) - f^{2n}(a)],$$

B_n being Bernoulli's nth number.

$$A \equiv \int_0^x f'(a+x)\,dx \equiv \int_0^x y\,dx, \text{ and } R_{2n} \equiv \frac{x^{2n}}{(2n)!}[f^{2n}(a+\theta_1 x) - f^{2n}(a+\theta_2 x)],$$

where both θ_1 and θ_2 have some value greater then zero and less than unity.

$A=\frac{1}{3}(4C_2-C_1) \pm \frac{4}{15}R_4$ . . . . (1).

$A=\frac{1}{8}(9C_3-C_1) \pm \frac{4}{45}R_4$ . . . . (2).

$A=\frac{1}{45}(64C_4-20C_2+C_1) \pm \frac{3}{70}R_6$ . . . . (3).

$A=\frac{1}{120}(243C_3-128C_2+5C_1) \pm \frac{2}{21}R_6$ . . . . . . (4).

$A=\frac{1}{2520}(8192C_4-6561C_3+896C_2-7C_1) \pm .0229R_8$ . . . . (5).

$A=\frac{1}{350}(512C_{12}-175C_6+14C_3-C_2) \pm .000{,}14R_8$ . . . . (6).

$A=\frac{1}{35}(56C_{12}-28C_6+8C_4-C_3) \pm .000{,}099R_8$ . . . . . (7).

$A=\frac{1}{10}(15C_6-6C_3+C_2) \pm .006{,}62R_6$ . . . . (8).

$A=\frac{1}{210}(567C_6-512C_4+162C_3-7C_2) \pm .002{,}55R_8$ . . . . (9).

$A=\frac{1}{4200}(10{,}368C_6-8{,}192C_4+2{,}025C_3-C_1) \pm .001{,}93R_8$ . . . (10).

$A = \frac{1}{3150}(6,561C_6 - 3,584C_4 + 175C_2 - 2C_1) \pm .003,77R_8$ . . . . (11).

$A = \frac{1}{840}(1,296C_6 - 567C_3 + 112C_2 - C_1) \pm .008,39R_8$ . . . . . (12).

$A = \frac{1}{12600}(34,992C_6 - 32,768C_4 + 10,935C_3 - 560C_2 + C_1) \pm .002,74R_{10}$ (13).

$A = \frac{1}{2835}(4,096C_8 - 1,344C_4 + 84C_2 - C_1) \pm .002,02R_8$ . . . . (14).

$A = \frac{1}{8731800}(33,554,432C_8 - 31,177,872C_6 + 7,569,408C_4 - 1,240,029C_3$
$+ 25,872C_2 - 11C_1) \pm .000,225R_{12}$ . . . . (15).

$A = \frac{1}{1801800}(2,985,984C_{12} - 1,667,952C_6 + 585,728C_4 - 104,247C_3$
$+ 2,288C_2 - C_1) \pm .000,095,6R_{12}$ . . . . (16).

$A = \frac{1}{1050}(1,728C_{12} - 945C_6 + 320C_4 - 54C_3 + C_2) \pm .000,071,2R_{10}$. (17).

$A = \frac{1}{280}(384C_6 - 105C_3 + C_1) \pm .006,05R_6$ . . . . (18).

$A = C_6 + \frac{13}{35}(C_6 - C_3) - \frac{1}{280}(C_3 - C_1) \pm .006,05R_6$ . . . . . (18a).

$A = \frac{1}{210}(243C_6 - 35C_2 + 2C_1) \pm .017,1R_6$ . . . . (19).

$A = C_6 + \frac{11}{70}(C_6 - C_2) - \frac{1}{105}(C_2 - C_1) \pm .017,1R_6$ . . . . . (19a).

$A = \frac{1}{45}(64C_8 - 20C_4 + C_2) \pm .001,34R_6$ . . . . (20).

$A = \frac{1}{722925}(1,048,576C_{16} - 348,160C_8 + 22,848C_4 - 340C_2 + C_1)$
$\pm .000,030,2R_{10}$ . . . . (21).

$A = \frac{1}{6527025000}(110,075,314,176C_{24} - 64,900,362,240C_{12} + 25,081,937,920C_8$
$- 5,178,990,960C_6 + 202,076,160C_4 - 9,808,695C_3 + 38,640C_2 - C_1)$
$\pm .000,000,134R_{16}$ . . . . (22).

$A = \frac{1}{2376}(3,200C_{10} - 825C_5 + C_1) \pm .000,624R_6$ . . . . . . (23).

$A = \frac{1}{2010093768}(3,648,810,176C_{18} - 1,674,098,987C_9 + 36,837,504C_6$
$- 575,586C_3 + 20,672C_2 - 11C_1) \pm .000,000,736R_{12}$ . . . . (24).

Formulae involving dy/dx.

$A = \frac{1}{15}(16C_{2,1} - C_{1,1}) \pm .115R_6$ . . . . (25).

$A = C_{2,1} + \frac{1}{15}(C_{2,1} - C_{1,1}) \pm .115R_6$ . . . . (25a).

$A = \frac{1}{80}(81C_{3,1} - C_{1,1}) \pm .019,1R_6$ . . . . (26).

$A = C_{3,1} + \frac{1}{80}(C_{3,1} - C_{1,1}) \pm .019,1R_6$ . . . . (26a).

$A = \frac{1}{945}(1,024C_{4,1} - 80C_{2,1} + C_{1,1}) \pm .005,9R_8$ . . . . (27).

$A = C_{4,1} + \frac{79}{945}(C_{4,1} - C_{2,1}) - \frac{1}{945}(C_{2,1} - C_{1,1}) \pm .005,9R_8$ . . . . . (27a).

$A = \frac{1}{1680}(2,187C_{3,1} - 512C_{2,1} + 5C_{1,1}) \pm .019,6R_8$ . . . . . . (28).

$A = \frac{1}{75600}(131,072C_{4,1} - 59,049C_{3,1} + 3,584C_{2,1} - 7C_{1,1}) \pm .002,88R_{10}$. (29).

$A = \frac{1}{33775}(36,864C_{12,1} - 3,150C_{6,1} + 63C_{3,1} - 2C_{2,1}) \pm .000,002,76R_{10}$. (30).

$$A=\tfrac{1}{245}(270C_{6,1}-27C_{3,1}+2C_{2,1})\pm.000{,}389R_{8} \qquad (31).$$

$$A=\tfrac{1}{6825}(10{,}206C_{6,1}-4{,}096C_{4,1}+729C_{3,1}-14C_{2,1})\pm.000{,}151R_{10} \qquad (32).$$

$$A=\tfrac{1}{260400}(373{,}248C_{6,1}-131{,}072C_{4,1}+18{,}225C_{3,1}-C_{1,1})\pm.000{,}118R_{10} \qquad (33).$$

$$A=\tfrac{1}{89775}(118{,}098C_{6,1}-28{,}672C_{4,1}+350C_{2,1}-C_{1,1})\pm.000{,}25R_{10} \qquad (34).$$

$$A=\tfrac{1}{42000}(46{,}656C_{6,1}-5{,}103C_{3,1}+448C_{2,1}-C_{1,1})\pm.000{,}633R_{10} \qquad (35).$$

$$A=\tfrac{1}{831600}(1{,}259{,}712C_{6,1}-524{,}288C_{4,1}+98{,}415C_{3,1}-2{,}240C_{2,1}+C_{1,1})$$
$$\pm.000{,}202R_{12} \qquad (36).$$

$$A=\tfrac{1}{240975}(262{,}144C_{8,1}-21{,}504C_{4,1}+336C_{2,1}-C_{1,1})\pm.000{,}089{,}8R_{10} \qquad (37).$$

$$A=\tfrac{1}{1135134000}(2{,}147{,}483{,}648C_{8,1}-1{,}122{,}403{,}392C_{6,1}+121{,}110{,}528C_{4,1}$$
$$-11{,}160{,}261C_{3,1}+103{,}488C_{2,1}-11C_{1,1})\pm.000{,}010{,}7R_{14} \qquad (38).$$

$$A=\tfrac{1}{378378000}(429{,}981{,}696C_{12,1}-60{,}046{,}272C_{6,1}+9{,}371{,}648C_{4,1}$$
$$-938{,}223C_{3,1}+9{,}152C_{2,1}-C_{1,1})\pm.000{,}002{,}8R_{14} \qquad (39).$$

$$A=\tfrac{1}{109725}(124{,}416C_{12,1}-17{,}010C_{6,1}+2{,}560C_{4,1}-243C_{3,1}+2C_{4,1})$$
$$\pm.000{,}001{,}66R_{12} \qquad (40).$$

$$A=\tfrac{1}{299376}(320{,}000C_{10,1}-20{,}625C_{5,1}+C_{1,1})\pm.000{,}014{,}3R_{8} \qquad (41).$$

Formulae involving $\frac{dy}{dx}$, $\frac{d^3y}{dx^3}$ and $\frac{d^5y}{dx^5}$.

$$A=\tfrac{1}{63}(64C_{2,2}-C_{1,2})\pm.078{,}4R_{8} \qquad (42).$$

$$A=C_{2,2}+\tfrac{1}{63}(C_{2,2}-C_{1,2})\pm.078{,}4R_{8} \qquad (42a).$$

$$A=\tfrac{1}{728}(729C_{3,2}-C_{1,2})\pm.006{,}03R_{8} \qquad (43).$$

$$A=C_{3,2}+\tfrac{1}{728}(C_{3,2}-C_{1,2})\pm.006{,}03R_{8} \qquad (43a).$$

$$A=\tfrac{1}{16065}(16384C_{4,2}-320C_{2,2}+C_{1,2})\pm.001{,}31R_{10} \qquad (44).$$

$$A=\tfrac{1}{17640}(19{,}683C_{3,2}-2{,}048C_{2,2}+5C_{1,2})\pm.007{,}05R_{10} \qquad (45).$$

$$A=C_{3,2}+\tfrac{227}{1960}(C_{3,2}-C_{2,2})-\tfrac{1}{3528}(C_{2,2}-C_{1,2})\pm.007{,}05R_{10} \qquad (45a).$$

$$A=\tfrac{1}{1580040}(2{,}097{,}152C_{4,2}-531{,}441C_{3,2}+14{,}336C_{2,2}-7C_{1,2})\pm.000{,}669R_{12} \qquad (46).$$

$$A=\tfrac{1}{5195575}(5{,}308{,}416C_{12,2}-113{,}400C_{6,2}+567C_{3,2}-8C_{2,2})$$
$$\pm.000{,}000{,}086{,}3R_{12} \qquad (47).$$

$$A=\tfrac{1}{9485}(9{,}720C_{6,2}-243C_{3,2}+8C_{2,2})\pm.000{,}037{,}9R_{10} \qquad (48).$$

$$A=C_{6,2}+\tfrac{47}{1897}(C_{6,2}-C_{3,2})-\tfrac{8}{9485}(C_{3,2}-C_{2,2})\pm.000{,}037{,}9R_{10} \qquad (48a).$$

$$A=\tfrac{1}{37837800}(45{,}349{,}632C_{6,2}-8{,}388{,}608C_{4,2}+885{,}735C_{3,2}-8{,}960C_{2,2}$$
$$+C_{1,2})\pm.000{,}027{,}3R_{14} \qquad (49).$$

$$A=\tfrac{1}{1635480}(1{,}679{,}616C_{6,2}-45{,}927C_{3,2}+1{,}792C_{2,2}-C_{1,2})\pm.000{,}079R_{12} \qquad (50).$$

$A=\frac{1}{164344495}(16{,}777{,}216C_{8,2}-344{,}064C_{4,2}+1{,}344C_{2,2}-C_{1,2})$
$\quad \pm.000{,}006{,}39R_{12}$ . . . . (51).

$A=\frac{1}{5989723937400}(61{,}917{,}364{,}224C_{12,2}-2{,}161{,}665{,}792C_{6,2}+149{,}946{,}368C_{4,2}$
$\quad -8{,}444{,}007C_{3,2}+36{,}608C_{2,2}-C_{1,2})\pm.000{,}000{,}108R_{16}$ . . (52).

$A=\frac{1}{255}(256C_{2,3}-C_{1,3})\pm.073{,}1R_{10}$ . . . . (53).

$A=C_{2,3}+\frac{1}{255}(C_{2,3}-C_{1,3})\pm.073{,}1R_{10}$ . . . . (53a).

$A=\frac{1}{6560}(6{,}561C_{3,3}-C_{1,3})\pm.002{,}53R_{10}$ . . . . (54).

$A=C_{3,3}+\frac{1}{6560}(C_{3,3}-C_{1,3})\pm.002{,}53R_{10}$ . . . . (54a).

$A=\frac{1}{260865}(262{,}144C_{4,3}-1{,}280C_{2,3}+C_{1,3})\pm.000{,}391R_{12}$ . . . . (55).

$A=C_{4,3}+\frac{1}{260865}[1{,}279(C_{4,3}-C_{2,3})-(C_{2,3}-C_{1,3})]\pm.000{,}391R_{12}$ . . (55a).

$A=\frac{1}{600600000}(60{,}466{,}176C_{6,3}-413{,}343C_{3,3}+7{,}168C_{2,3}-C_{1,3})$
$\quad \pm.000{,}013{,}3R_{14}$ . . . . (56).

Formulae expressed in terms of ordinates.

$A=\frac{x}{6}[(y_0+y_2)+4y_1]\pm\frac{4}{15}R_4$ (Simpson's first formula) . . (1r).

$A=\frac{x}{8}[(y_0+y_3)+3(y_1+y_2)]\pm\frac{4}{45}R_4$ (Simpson's second formula) . (2r).

$A=\frac{x}{90}[7(y_0+y_4)+32(y_1+y_3)+12y_2]\pm\frac{3}{70}R_6$ . . . . . . (3r).

$A=\frac{x}{120}[11(y_0+y_6)+81(y_2+y_4)-64y_3]\pm\frac{2}{21}R_6$ . . . . . (4r).*

$A=\frac{x}{2520}[151(y_0+y_{12})+2{,}048(y_3+y_9)+2{,}496y_6-2{,}187(y_4+y_8)$
$\quad \pm.022{,}9R_8$ . . . . (5r).*

$A=\frac{x}{2100}[53(y_0+y_{12})+256(y_1+y_3+y_5+y_7+y_9+y_{11})+81(y_2+y_{10})$
$\quad +78y_6+109(y_4+y_8)]\pm.000{,}14R_8$ . . . . (6r).

$A=\frac{x}{210}[5(y_0+y_{12})+28(y_1+y_5+y_7+y_{11})+40(y_3+y_9)+12y_6$
$\quad -2(y_4+y_8)]\pm.000{,}099R_8$ . . . . (7r).

$A=\frac{x}{20}[(y_0+y_2+y_4+y_6)+5(y_1+y_5)+6y_3]\pm.006{,}62R_6$ (Weddle's
formula) . . . . (8r).

$A=\frac{x}{420}[17(y_0+y_{12})+189(y_2+y_{10})+297(y_4+y_8)-256(y_3+y_9)-74y_6]$
$\quad \pm.002{,}55R_8$ . . . . (9r).*

*Note that ordinates are not all equidistant, but would be so if certain missing ordinates, as indicated by the subscripts, be supplied.

$$A=\frac{x}{5670}[217(y_0+y_8)+1{,}024(y_1+y_3+y_5+y_7)+352(y_2+y_6)+436y_4] \pm .002{,}02R_8 \quad \text{(14r).}$$

$$A=\frac{x}{840}[41(y_0+y_6)+216(y_1+y_5)+27(y_2+y_4)+272y_3] \pm .008{,}39R_8 \quad \text{(12r).}$$

$$A=\frac{x}{280}[15(y_0+y_6)+64(y_1+y_3+y_5)+29(y_2+y_4)] \pm .006{,}05R_6 \quad \text{(18r).}$$

$$A=\frac{x}{420}[25(y_0+y_6)+81(y_1+y_2+y_4+y_5)+46y_3] \pm .017{,}1R_6 \quad \text{(19r).}$$

$$A=\frac{x}{2100}[49(y_0+y_{12})+288(y_1+y_5+y_7+y_{11})+448(y_3+y_9)-27(y_2+y_{10}) -63(y_4+y_8)+134y_6] \pm .000{,}071{,}2R_{10} \quad \text{(17r).}$$

$$A=\frac{x}{1801800}[41{,}833(y_0+y_{12})+248{,}832(y_1+y_5+y_7+y_{11})+395{,}264(y_3+y_9) +118{,}416y_6-29{,}160(y_2+y_{10})-63{,}909(y_4+y_8)] \pm .000{,}095{,}6R_{12} \quad \text{(16r).}$$

Formulae expressed in terms of ordinates and the derivatives of the extreme ordinates, wherein $E_n \equiv [f^n(a+x)-f^n(a)]$.

$$A=\frac{x}{30}[7(y_0+y_2)+16y_1]-\frac{x^2}{60}E_2 \pm .115R_6 \quad \text{(25s).}$$

$$A=\frac{x}{80}[13(y_0+y_3)+27(y_1+y_2)]-\frac{x^2}{120}E_2 \pm .019{,}1R_6 \quad \text{(26s).}$$

$$A=\frac{x}{1890}[217(y_0+y_4)+512(y_1+y_3)+432y_2]-\frac{x^2}{252}E_2 \pm .005{,}9R_8 \quad \text{(27s).}$$

$$A=\frac{x}{1680}[239(y_0+y_6)+729(y_2+y_4)-256y_3]-\frac{x^2}{168}E_2 \pm .019{,}6R_8 \quad \text{(28s).*}$$

$$A=\frac{x}{75600}[7{,}435(y_0+y_{12})+32{,}768(y_3+y_9)+34{,}560y_6-19{,}683(y_4+y_8)] -\frac{x^2}{360}E_2 \pm .002{,}88R_{10} \quad \text{(29s).*}$$

$$A=\frac{x}{13650}[913(y_0+y_{12})+3{,}402(y_2+y_{10})-2{,}048(y_3+y_9)+3{,}888(y_4+y_8) +1{,}340y_6]-\frac{x^2}{780}E_2 \pm .000{,}151R_{10} \quad \text{(32s).*}$$

$$A=\frac{x}{42000}[3{,}149(y_0+y_6)+7{,}776(y_1+y_5)+6{,}075(y_2+y_4)+8{,}000y_3] -\frac{x^2}{600}E_2 \pm .000{,}633R_{10} \quad \text{(35s).}$$

$$A=\frac{x}{378378000}[13{,}929{,}421(y_0+y_{12})+35{,}831{,}808(y_1+y_5+y_7+y_{11}) +25{,}824{,}096(y_2+y_{10})+38{,}174{,}720(y_3+y_9)+25{,}511{,}355(y_4+y_8)$$

*Note that ordinates are not all equidistant, but would be so if certain missing ordinates, as indicated by the subscripts, be supplied.

$$+28,171,584y_6] - \frac{x^2}{2520}E_2 \pm .000,002,8R_{14} \quad . \quad . \quad . \quad . \quad . \quad (39s).$$

$$A = \frac{x}{126}[31(y_0+y_2)+64y_1] - \frac{5x^2}{252}E_2 + \frac{x^4}{15120}E_4 \pm .078,4R_8 \quad . \quad . \quad (42t).$$

$$A = \frac{x}{728}[121(y_0+y_3)+243(y_1+y_2)] - \frac{5x^2}{546}E_2 + \frac{x^4}{65520}E_4 \pm .006,03R_8 \quad (43t).$$

$$A = \frac{x}{32130}[3,937(y_0+y_4)+8,192(y_1+y_3)+7,872y_2] - \frac{x^2}{204}E_2 + \frac{x^4}{257040}E_4$$
$$\pm .001,31R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44t).$$

$$A = \frac{x}{1635480}[132,761(y_0+y_6)+279,936(y_1+y_5)+264,627(y_2+y_4)$$
$$+280,832y_3] - \frac{25x^2}{11682}E_2 + \frac{x^4}{1401840}E_4 \pm .000,079R_{12} \quad . \quad . \quad (50t).$$

The well known *Simpson's rule* and *Weddle's rule* involve the successive use of *Simpson's formula* and *Weddle's formula* respectively. (See formulae (1*r*), (2*r*) and (8*r*) above). Similar use may, of course, be made of other of the foregoing formulae in y.

Still other formulae may be had in these ordinates by eliminating between two or more equations involving the same set, as between (12*r*) and (18*r*). The new formulae thus formed are usually less accurate than those above given, having been formed by considering the remainders zero or negligible, but may be advantageously used in cases where the values of certain ordinates are not readily obtainable.

For ready reference the numerical coefficients in some of the foregoing formulae and their logarithms are given below in decimal form. There may be an error of ± 1 in the last decimal place, as results were obtained by using a 10-place table of logarithms.

Formula (15)

C_8	3.842,785,222	.584,646,111,9
C_6	3.570,612,245	.552,742,690,0
C_4	.866,878,308	9.937,958,135,2—10
C_3	.142,012,987	9.152,328,062,0—10
C_2	.002,962,963	7.471,726,222,9—10
C_1	.000,001,260	4.100,288,905,5—10

Formula (21)

C_{16}	1.450,463,049	.161,506,669,3
C_8	.481,599,060	9.682,685,630,1—10
C_4	.031,604,938	8.499,754,946,5—10
C_2	.000,470,312	6.672,385,673,4—10
C_1	.000,001,383	4.140,906,756,4—10

Formula (22)

C_{24}	1.686,455,775	.226,974,956,5
C_{12}	.994,333,666	9.997,532,144,1−10
C_8	.384,278,522	9.584,646,111,9—10
C_6	.079,346,939	8.899,530,176,2—10
C_4	.003,095,994	7.490,800,103,7−10
C_3	.000,150,278	6.176,896,253,7—10
C_2	.000,000,592	3.772,322,141,0−10
C_1	.000,000,000	9.185,285,023,3−20

Formula (24)

C_{18}	1.814,431,370	.258,740,545,7
C_9	.832,473,483	9.920,370,408,9—10
C_6	.018,318,060	8.262,879,471,3−10
C_3	.000,286,220	6.456,699,497,3—10
C_2	.000,010,279	5.011,971,771,6—10
C_1	.000,000,005	1.737,981,960,5—10

Formula (39)

$C_{12,1}$	1.136,381,334	.055,524,091,1
$C_{6,1}$	.158,693,878	9.200,560,171,9−10
$C_{4,1}$	.024,767,952	8.393,890,090,8−10
$C_{3,1}$	.002,479,592	7.394,380,197,9—10
$C_{2,1}$	.000,024,187	5.383,590,134,1−10
$C_{1,1}$	.000,000,003	1.422,074,122,7−10

Formula (50)

$C_{6,2}$	1.026,986,573	.011,564,765,5
$C_{3,2}$	.028,081,664	8.448,422,840,3—10
$C_{2,2}$	.001,095,703	7.039,692,767,8−10
$C_{1,2}$	.000,000,611	3.786,354,762,5—10

Formula (51)

$C_{8,2}$	1.020,853,761	.008,963,532,4
$C_{4,2}$	.020,935,477	8.320,882,870,6—10
$C_{2,2}$	.000,081,779	5.912,642,905,3−10
$C_{1,2}$	.000,000,061	2.784,243,636,6−10

Formula (52)

$C_{12,2}$	1.033,726,545	.014,405,668,3
$C_{6,2}$	.036,089,574	8.557,381,757,7−10
$C_{4,2}$	.002,503,394	7.398,529,158,3−10
$C_{3,2}$	.000,140,975	6.149,141,792,4—10
$C_{2,2}$	.000,000,611	3.786,169,210,6−10
$C_{1,2}$	.000,000,000	9.222,593,207,8−20

EXAMPLES.

Ex. 1. Construct a five-place table of the values of $\int_0^{n\pi}\frac{\sin x}{x}$ for all integral values of n from 1 to 10, inclusive.

Solution. $\int_0^{10\pi}\frac{\sin x}{x}dx=\sum_{n=0}^{n=10}\int_0^{\pi}\frac{\sin(n\pi+x)}{n\pi+x}dx.$ Similarly for $n=9$, 8, 7, etc.

Each value to be found may therefore be derived by the aid of the sum of all the preceding values, already found.

To determine which of the given formulae is sufficiently accurate and at the same time involves the minimum amount of labor is now advisable. Formula (17) has $\pm.0000712R_{10}$ for its remainder term, and, as seems probable, ought to be within the bounds of accuracy required. Expressed more fully this remainder term is

$$\pm\frac{712}{10^7}\cdot\frac{\pi^{10}}{10!}[f^{10}(x_1)-f^{10}(x_2)],\text{ where } 0\lesseqgtr x_1\lesseqgtr\pi \text{ and } 0\lesseqgtr x_2\lesseqgtr\pi.$$

Developing $\sin x/x$ into a power series in x, and differentiating nine times to obtain the tenth derivative, we get another power series in x, the value of which lies somewhere between 0 and $-\frac{1}{5}$ for $0\lesseqgtr x\lesseqgtr\pi$. Substituting $-\frac{1}{5}$ for the bracket expression above, we find that the remainder term has a value within the limits of $\pm.000,000,4$. (17) is therefore sufficiently accurate for this value of x, and it can be readily shown that its value for other values of x involved in the problem is less than its value between the limits here considered.

The chordal areas (represented generally by C with a subscript equal to the number of equidistant ordinates employed, less one, the first ordinate being taken at the point $x=0$ and the last at the point $x=X$, where X is the maximum value of x), involved in (17) are,

$$C_{12}=1.850{,}119,\quad C_6=1.844{,}651,\quad C_4=1.835{,}508,$$
$$C_3=1.822{,}636,\text{ and } C_2=1.785{,}398.$$

Substituting in (17), $\int_0^{\pi}\frac{\sin x}{x}dx=1.851{,}937.$

To check the accuracy of the numerical work done, these values of C may be substituted in (8), and, by doing so, it is found that the same result is obtained by either formula. It is therefore probably safe to employ the simpler formula in computing the remaining terms sought.

It is also instructive to apply some of the other formulae involving

the same values of C, *e. g.*, (9), (10), (11), (13), (16), etc., as well as those involving the second, fourth and sixth derivatives of the function sought, such as (43a), (45a), (48a), (55a), etc.

Ex. 2. Find the common logarithm of 2 from the identity,

$$10 \log 2 - \log 1000 = M \int_0^{.024} \frac{dx}{1+x},$$

M being the modulus of the common system of logarithms.

Here the remainder term is easily evaluated, and the selection of the formula to be used is determined largely by the degree of accuracy required. Employing (35s) we get, log2=.301,029,995,663,981,195,213,738,269,3, which we know from the form of the remainder must be correct to not less than 21 decimal places. Expressed in terms of the formula after some simple reductions have been made, the development is,

$$\log 2 = \tfrac{3}{10} + \frac{M}{10}\left(\frac{253}{3,500}\left[\frac{3,149}{64,000} + \frac{7,776}{64,005} + \frac{6,075}{64,008}\right] + \frac{8}{1,771} - \frac{2,277}{10^8.512}\right)$$
$$\pm .000,633 R_{10}.$$

Using (39s) a result correct to thirty places of decimals would have been assured. Should still greater accuracy be desired, the step-by-step process of Ex. 1 may be used.

Ex. 3. Similarly, the logarithm of 2 known, the common logarithm of 3, correct to forty-seven decimal places, may be obtained by (39s) from the identity,

$$8 \log 3 + \log 10 - 16 \log 2 = M \int_0^{\frac{37}{32768}} \frac{dx}{1+x}.$$

Ex. 4. Obtain the value of $\int_0^{\frac{1}{2}\pi} \frac{x\,dx}{\sin x(1+\frac{4}{25}\cos^2 x)^{\frac{3}{2}}}$ correct to six decimal places.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

364. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The English physicist, Hooke, published the discovery contained in the Latin sentence, "Ut tensio sic vis" by the cypher *ceiiinosssttuv*. Preserving the lexicographical order, find which permutation, taking all letters, the Latin sentence is from the cypher.

Solution by H. C. FEEMSTER, A. M., York College, York, Nebraska.

Beginning at the first, and counting all permutations, although letters may be alike, the permutation required is:

$$12|\underline{13}+10|\underline{12}+|\underline{10}+4|\underline{9}+5|\underline{8}+|\underline{7}+3|\underline{6}+3|\underline{5}+|\underline{4}+2|\underline{2}+|\underline{1}$$
$$=79,\ 519,\ 555,\ 109.$$

370. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Prove that, if the fraction m/p (p prime) gives a recurring decimal with an even number of digits in the cycle, the sum of the two halves will be composed of 9's. (The special case where the number$=p-1$ was proposed in the MONTHLY, Vol. IV, and answered in Vol. V, p. 11. The proof there given, however, is not complete.) The above property is true of other fractions, *e. g.*, $\frac{1}{77}=.\dot{0}1298\dot{7}$, $\frac{1}{133}=\frac{1}{7.19}=.\dot{0}07518796,99248120\dot{3}$. Find for what fractions this is true.

I. Solution by F. H. SAFFORD, Ph. D., University of Pennsylvania.

1°. To prove that the sum of the two halves will be composed of 9's. Let the number of digits in the cycle be $2n$, then the number of 9's in the denominator of the unreduced common fraction will be $2n$, and this sequence is divisible by the sequence of n 9's, *e. g.*, 999999 is divisible by 999. It follows that the unreduced numerator must be also divisible by the sequence of n 9's. Suppose the original cycle is $\dot{a}bcde\dot{f}$, the fraction may be written

$$\frac{(abc)(1000)+(def)}{999999}, \text{ or } \frac{(abc)(999)+(abc)+(def)}{999999},$$

in which (abc), etc., represents a number of three digits and not a product. The part $(abc)+(def)$ must be divisible by 999 as shown by the preceding.

Not all of the digits a, b, c, d, e, f, can be 9's, so that the sum is less than twice 999 and is thus 999. Hence for a given (abc) the corresponding (def) is fixed and has the specified property. The proof is extensible to any case and does not require that the recurring decimal be perfect.

2°. The results of the indicated cancellations are:

$$\frac{a+1}{11},\quad \frac{(ab)+1}{101},\quad \frac{(abc)+1}{1001},\ \text{etc.}$$

Of such fractions, those denominators (except the first) having an even number of digits are composite, also others such as $10001=73.137$. The numerators $(abc...)+1$ are unrestricted as might be anticipated, so that fractions having denominators of the form 10^n+1 have the required property. The illustration $\frac{1}{77}=.\dot{0}1298\dot{7}$ came from $\frac{012+1}{1001}$; the fraction $\frac{1}{133}$ originally had the numerator $007518796+1$ and the denominator 10^9+1.

II. **Solution by the PROPOSER.**

Let the number of digits in the cycle be $2n$. Let d be the value of the recurring decimal.

$d=.abc...(n \text{ digits})...\alpha\beta\gamma...(n \text{ digits})...$

Then $10^{2n}d=abc...\alpha\beta\gamma...\ .abc...\alpha\beta\gamma...$; $d=.abc...\alpha\beta\gamma...$

Subtracting, we have, $(10^{2n}-1)d=abc...\alpha\beta\gamma...$ an integer.

$\therefore d=\frac{abc...\alpha\beta\gamma...}{10^{2n}-1}=\frac{m}{p}$ by hypothesis.

Then $\frac{m(10^{2n}-1)}{p}=\frac{m(10^n-1)(10^n+1)}{p}$ is an integer$=abc...\alpha\beta\gamma...$

p cannot divide 10^n-1 since in that case the period of m/p would be n and not $2n$. Therefore p is a factor of 10^n+1.

$$\begin{array}{rl} 10^n(m/p)= & abc...\ .\alpha\beta\gamma...abc... \\ (m/p)= & \qquad .abc...\alpha\beta\gamma... \\ \hline (10^n+1)(m/p)= & abc...\ =\text{an integer by proof just given.} \end{array}$$

Therefore, the sum of the two decimals $.\alpha\beta\gamma...$ and $.abc...$ must be $.999...$

The answer to the second part of the question follows at once from the above. It is sufficient if p instead of being a prime is *any* factor of 10^n+1. $10^3+1=7.11.13$. Therefore if $p=77$, 91, 143, or 1001, a/p will have six figures in the cycle and the two halves will be complements.

III. **Solution by C. A. LAISANT, Paris, France.**

If the fraction m/pp' gives a periodical decimal fraction with an even number $2n$ of digits, and if we write the period $A.10^n+B$, we have

$$\frac{m}{pp'}=\frac{A.10^n+B}{(10^n-1)(10^n+1)}.$$

10^n-1 cannot be mult. p and mult. p', because the period would then have n digits, instead of $2n$.

Suppose 10^n+1 mult. p and mult. p', hence mult. pp' (supposing p, p' primes together). Writing $10^n+2=qpp'$, we have

$$mq=\frac{A.10^n+B}{10^n-1}=A+\frac{A+B}{10^n-1}.$$

But $A<10^n$, $B<10^n$. Then, $A+B=10^n-1$, and the property is true.

If 10^n+1=mult. p, and if p', divisor of 10^n-1, is 3 or 9, the sum $A+B$ is a number with similar digits.

Ex. $\frac{1}{21}=\frac{1}{7.3}=0{,}047619...$ 047+619=666.

$\frac{1}{63}=\frac{1}{7.9}=0{,}015373...$ 015+373=888.

Also solved by J. Scheffer.

H. C. Feemster should have received credit for solving 368.

GEOMETRY.

390. Proposed by PROF. R. C. ARCHIBALD, Brown University, Providence, R. I.

Find, geometrically, and without introducing focal properties, the locus of the vertices of the conjugate parallelograms of an ellipse.

Solution by the PROPOSER.

The solution which I had in mind when I proposed this problem was the following: By the elementary ideas of a *stretch* in projective geometry, if the radius of a circle, α, be b, and the stretch ratio be $\frac{a}{b}$ ($a>b$, say), α will be transformed into an ellipse with semi-axes of lengths a and b. Corresponding to any conjugate parallelogram of the ellipse will be a square circumscribing α; and since the locus of the vertices of all such squares is another circle concentric with α and of radius $\sqrt{(2)}b$, the stretch which transforms α into the ellipse with axes a and b will transform the concentric circle with radius $\sqrt{(2)}b$ into the ellipse with axes $\sqrt{(2)}a$, $\sqrt{(2)}b$. And this is the required locus, as found *analytically* by Mr. Scheffer (AMERICAN MATHEMATICAL MONTHLY, Vol. XIX, p. 54, March, 1912).

398. **Proposed by C. N. SCHMALL, New York City.**

In a square $ABCD$ draw the diagonal AC. Now bisect AD in G and draw GB cutting AC in H. Prove that $\triangle AGH=\frac{1}{2}\triangle CGH=\frac{1}{3}\triangle ABG=\frac{1}{4}\triangle BCH$.

Solution by A. H. HOLMES, Brunswick, Maine, and G. I. HOPKINS, A. M., Manchester, New Hampshire.

Since $BC=2AG$ by construction, and $\angle CBH=\angle GAH$, $CH=2AH$. But the altitudes of triangles AGH and CGH are the same.

$\therefore \triangle AGH=\frac{1}{2}\triangle CGH$.

For the same reason as above, $BH=2GH$.

$\therefore BG=3GH$.

The altitudes of triangles AGH and ABG being the same, $\triangle AGH=\frac{1}{3}\triangle ABG$. Since the homologous sides in triaugles AGH and BCH are one-half the size in the former of those in the latter, $\triangle AGH=\frac{1}{4}\triangle BCH$.

Also solved by J. Scheffer, Francis Rust, M. A. Muzzy, and H. C. Feemster.
A. M. Harding should have received credit for solving 397.

399. **Proposed by J. K. ELLWOOD, Superintendent of Schools, Lucas, Kansas.**

A race track is to be composed of two tangents and the arc of the circle which is concave towards the point of intersection of the two tangents, each tangent and the arc of the circle being 1 mile. What is the radius of the circle?

Solution by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, Ohio.

Let AT and BT be the tangents, and ACB the arc composing the track; O the center, and OA the radius of the circle.

Let $AOT=\theta$, and $AO=r$. Then arc $ACB=(2\pi-2\theta)r=1$, and $\tan\theta=1/r$; whence $\tan\theta=2(\pi-\theta)$.

This equation solved by approximation gives

$$\theta=74^\circ\ 46'.18. \qquad r=\frac{1\text{ mile}}{\tan\theta}=\frac{5280\text{ feet}}{\tan\theta}=1437.45\text{ feet.}$$

Also solved by J. E. Sanders, A. M. Harding, J. Scheffer, H. Prime, A. H. Holmes, Elmer Schuyler, E. B. Escott, and H. C. Feemster.

CALCULUS.

321. **Proposed by ARTEMAS MARTIN, Ph. D., LL. D., United States Coast and Geodetic Survey Office, Washington, D. C.**

To a person in a boat at the center of a circular pond the bottom appears to be perfectly level. What is the actual form of the bottom of the pond, the depth of the water at the center being a feet, and the distance of the eyes of the observer from the surface of the water being b feet. [From the *Mathematical Visitor*, Vol. 2, No. 2, p. 62.]

Solution by E. B. ESCOTT, Ann Arbor, Michigan.

Let A be the bottom of the pond at the center, OX the surface of the water, B the observer. Then $\sin\theta=m\sin\phi$.

Let $OQ=q$, $\sin\theta=\dfrac{q}{\sqrt{(q^2+b^2)}}$, $\sin\phi=\dfrac{q}{m\sqrt{(q^2+b^2)}}$;

$$QP=QR=\frac{a}{b}.\ BQ=\frac{a}{b}\sqrt{(q^2+b^2)};\ x=q+\frac{aq}{bm};\ y=QP\cos\phi$$

$$=a\sqrt{1+\frac{(m^2-1)x^2}{(a+bm)^2}};\ i.\ e.,\ \frac{y^2}{a^2}-\frac{x^2}{\dfrac{(a+bm)^2}{m^2-1}}=1,$$

hyperbola, center at O, AB principal axis.

An excellent solution was also received from H. C. Feemster.
This problem properly belongs to the Miscellaneous Department.

322. Proposed by E. B. ESCOTT, University of Michigan.

Find the equation of the curve such that the solid of revolution generated by revolving it about the x-axis shall have a volume equal to the m/nth part of the volume of the circumscribed cylinder.

Solution by H. PRIME, Boston, Massachusetts, and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let $\pi\int y^2dx=\pi\ (m/n)y^2x$.

Differentiating and arranging, $2m\ dy/y=(n-m)\ dx/x$.

Hence by integrating, $2m\log y=(n-m)\log x+\log c$.

$\log y^{2m}=\log x^{n-m}+\log c$, $y^{2m}=a^mx^{n-m}$ and $y^2=ax^{n/m-1}$.

This is the equation of the family of parabolas, including the case when $m/n=1/3$ and $y^2=ax^2$. Thus we have the familiar relation of the cone and its circumscribing cylinder.

Also solved similarly by M. A. Muzzy, and Elijah Swift.

323. Proposed by C. N. SCHMALL, New York City.

From what height must an elastic ball be dropped in order that, after impact with the hard surface of the sidewalk, it may rebound to a given altitude a in the least possible time from the moment of descent?

Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Let x be the height. Then will the velocity at the moment the ball strikes the surface$=\sqrt{(2gx)}$, and the time, t_1, $=\sqrt{(2x/g)}$. It rebounds now with a velocity$=e\sqrt{(2gx)}$, e being the coefficient of elasticity. Denoting the time it reaches the height a by t_2, we have $a=e\sqrt{(2gx)}.t_2-(gt_2^2/2)$, whence $t_2=\sqrt{(2/g)}]e\sqrt{x}\pm\sqrt{(e^2x-a)}]$. Consequently the total time is

$$T=t_1+t_2=\sqrt{\frac{2x}{g}}+\sqrt{\frac{2}{g}}[e\sqrt{x}\pm\sqrt{(e^2x-a)}].$$

Putting $dT/dx=0$, we get $x=\dfrac{a(1+e)^2}{e^2(1+2e)}$.

Also solved by C. N. Schmall, and H. Prime. In their solutions, these gentlemen use $e=1$.

324. Proposed by V. M. SPUNAR, Chicago, Ill.

A parabola slides between two rectangular axes; find (a) the locus of the focus, and (b) the locus of the vertex.

I. Solution by ELIJAH SWIFT, Princeton University.

Take the parabola in the form $y^2=4ax$. The equation of any two perpendicular tangents may be written $y=mx+(a/m)$ and $y=-(1/m)x-am$. The distances of these from the focus are $Y=(a/m)\sqrt{(1+m^2)}$ and $X=a\sqrt{(1+m^2)}$, and these quantities would therefore be the coördinates of the focus of a moving parabola, if the two perpendicular tangents were axes. Eliminating m, we get as the locus

$$x^2y^2=a^2(x^2+y^2), \text{ or in polar coördinates, } r=\frac{2a}{\sin 2\phi}.$$

In case (b), $Y=\dfrac{a}{m\sqrt{(1+m^2)}}$, $X=\dfrac{am^2}{\sqrt{(1+m^2)}}$.
Elimination of m gives us $x^2y^2(x^{\frac{2}{3}}+y^{\frac{2}{3}})^3=a^6$.

261. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

A man six feet high, walking at a rate of 100 yards a minute, crosses a muddy road close behind a wheel of a carriage which is going thrice as fast and in a direction at right angles to that of the man's motion. The diameter of the wheel is five feet. If, when the man is four feet from the middle of the wheel the mud is splashed up to the hight of seven feet, will any of it touch him? Unsolved in *Educational Times*.

Solution by H. PRIME, Boston, Massachusetts.

Let ϕ be the angle of elevation, v the velocity, of particles of mud leaving the ground at A; let the line ABH be the path of the wheel, $DEFGHI$ the man's path, the man being at D when the wheel touches A, at F when it touches B, and the mud is 7 feet high. Then $BF=4$ feet, $BH=3FH-\frac{5}{2}$, $16-FH^2=(3FH-\frac{5}{2})^2$, $FH=1.989960$, $BH=3.469879$.

The man is at three critical points in his path,—at E when the mud is 6 feet high rising, at G when it is 6 feet high falling, at I when the mud is striking the ground. While the man is between E and G, no mud can touch him, for it is more than 6 feet high. Between D and E only rising mud can hit him, between G and I only falling mud.

If $g=32$, the mud rises 7 feet, or falls 7 feet, in $\sqrt{\frac{7}{16}}$ seconds, and rises the last foot, or falls the first foot, in $\frac{1}{4}$ second. While the mud is rising, the carriage goes from A to B, 15 feet a second. Hence $AB=15\sqrt{\frac{7}{16}}=9.921568$, $AH=13.391447$; also $EF=FG=1.25$, $DF=FI=3.307189$; $EH=3.239560$, $GH=0.739560$, $HI=1.317630$; hence $AE=13.778007$, $AG=13.412149$, $AI=13.456409$.

For determining the limiting values of ϕ or v we have the equation of the parabola in which a projected particle of mud moves, A being the origin, $y=x\tan\phi-gx^2/2v^2\cos^2\phi$; also the maximum height$=v^2\sin^2\phi/2g=7$. Whence $\tan\phi=[14\pm\sqrt{(196-28y)}]/x$.

At E, $x=AE$, $y=6$. Hence $\phi=32^\circ\ 17'\ 43''$ or $v=39.6158$ feet a second.
At G, $x=AG$, $y=6$. Hence $\phi=55^\circ\ 11'\ 30''$ or $v=25.7787$ feet a second.
At I, $x=AI$, $y=0$. Hence $\phi=64^\circ\ 19'\ 54''$ or $v=23.5149$ feet a second.

Thus, if $32^\circ\ 17'\ 43''<\phi<55^\circ\ 11'\ 30''$, or if $\phi>64^\circ\ 19'\ 54''$, none of the mud touches the man. For in the former case it passes over his head, in the latter it does not reach to him. The man is hit between D and E if $\phi<32^\circ\ 17'\ 43''$, or between G and I if $55^\circ\ 11'\ 30''<\phi<64^\circ\ 19'\ 54''$.

Also solved by the Proposer.

262. Proposed by **V. M. SPUNAR, M. and E. E., Chicago, Ill.**

A hemispherical shell, whose radius is equal to the mean radius of the earth and whose thickness is one centimeter, is constructed of a matter whose density is equal to the mean density of the earth. A particle starts from rest at the center of the shell under the action of the attraction of the shell. Express as the decimal of a year the time it takes the particle to reach the surface of the shell, and find the velocity in centimeters per second of the particle just before it reaches the shell. Unsolved in *Educational Times.*

I. **Solution by the PROPOSER.**

$\dfrac{\text{The total volume of the earth}}{\text{The volume of the shell}}=\frac{2}{3}R$, nearly; $R=\dfrac{40,000,000}{2\pi}m=\dfrac{2}{\pi}10^9cm$, the mean radius of the earth.

$\therefore\ \dfrac{\text{The mass of the earth, } M'}{\text{The mass of the shell, } M}=\dfrac{147\times10^9R^2}{M}=\frac{2}{3}R$, at the same mean density of the earth.

$\therefore M=\frac{3}{2}\times147\times10^9Rkg$ (R in meters)$=\dfrac{441\times10^{19}}{\pi}g$.

The center of gravity, O, of the mass M is the center of gravity of the circular arc on the axis of symmetry (X-axis) at the distance $(2/\pi)R$ from O', the center of the shell.

The force of attraction, F, exerted by O on the particle (mass=1) in any position P between $O'O$ is CM/x^2, where x=abscissa of P and C, the constant of gravitation$=6.5\times10^{-8}$ (Baily). C becomes, however, 1 if cm.—sec. would be chosen and 1.537×10^7g as unity of the mass, M. Hence, $MC=\dfrac{441\times10^{15}}{1537\pi}$. Then,

$$\int_{v_0}^{v_1} v\,dv = \int_x^{2R/\pi} F\,dx = MC\int_x^{2R/\pi} \frac{dx}{x^2}. \quad \therefore \tfrac{1}{2}(v_1^2 - v_0^2) = \frac{MC(2R-\pi x)}{2xR}.$$

$$\therefore v_1^2 = \frac{MC(2R-\pi x)}{Rx} + v_0^2 = \left(\frac{dx}{dt}\right)^2,$$

where the initial velocity $v_0=0$.

It is evident, by the last formula, that the motion of the particle is such that if it reaches the point $x_1=2R/\pi$ $(dx/dt=0)$ will turn there and move in the other direction with its maximum velocity at O (the latter not necessarily $=\infty$ as the formula shows, which is only true for $0<x\leq x_1$).

$$\therefore t_1 = \sqrt{\frac{R}{MC}}\int_0^{2R/\pi} \frac{\sqrt{[(x)dx]}}{\sqrt{[2R-\pi x]}} = \sqrt{\left(\frac{R^3}{MC\pi}\right)} = \sqrt{\left(\frac{8\times 1537\times 10^{12}}{441\,\pi^3}\right)}$$

$=948,283$ seconds $=10$ days, 23 hours, 24 minutes, 42 seconds, is the time the particle takes to reach the center of gravity; and that, the particle takes to move from O to the shell itself,

$$t_2 = \sqrt{\left[\left(\frac{R}{\pi}\right)^3 \frac{1}{MC}\right]}\{\cos^{-1}(\pi-3) - \sqrt{[(\pi-2)(4-\pi)]}\}$$

$=259,916$ seconds $=3$ days, 0 hours, 11 minutes, 56 seconds.

Therefore, the total time in question, $T=t_1+t_2=1,208,199$ seconds $=$ 13 days, 23 hours, 36 minutes, 38 seconds $=\dfrac{1,208,199}{31,556,936}=.038286+$ year.

From symmetry of the motion we infer that the velocity, v_2, of the particle just before reaching the shell is the same (with respect to its scalar value) as that at the point $x_2=R(\pi-2)/\pi$.

$$\therefore v_2 = \sqrt{\left(\frac{\pi\, MC(4-\pi)}{R(\pi-2)}\right)} = 1000\sqrt{\left(\frac{441(4-\pi)\pi}{1537(\pi-2)2}\right)} = 582.15 \text{ cm/sec.}$$

II. **Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, New York.**

The component of the attraction along the axis of symmetry at any point on this axis within the shell is

$$X = \frac{\pi\rho r}{c^2}\left[\frac{r^2 - c^2 + y^2}{y}\right]_{y_0}^{y_1}$$

[Pierce, *Newtonian Potential Functions*, page 12], where ρ=mass of 1cc. of

matter, r is the radius of the sphere, c is the distance of the point from the center, and y is the distance from the point to the element of area. The limits are $y_0=\sqrt{[r^2+c^2]}$, $y_1=r-c$.

$$X=\frac{\pi\rho r}{c^2}\left[\frac{r^2-c^2+(r-c)^2}{r-c}-\frac{r^2-c^2+r^2+c^2}{\sqrt{[r^2+c^2]}}\right]=\frac{2\pi\rho r^2}{c^2}\left[1-\frac{r}{\sqrt{[r^2+c^2]}}\right].$$

$$X=\frac{d^2c}{dt^2}.\quad 2\frac{dc}{dt}\frac{d^2c}{dt^2}=4\pi\rho r^2\left[\frac{1}{c^2}-\frac{r}{c^2\sqrt{[r^2+c^2]}}\right]\frac{dc}{dt}.$$

$$\left(\frac{dc}{dt}\right)^2=v^2=4\pi\rho r^2\left[-\frac{1}{c}+\frac{\sqrt{[r^2+c^2]}}{rc}\right]+k_1.$$

$v=0$ when $c=0$. $-\dfrac{1}{c}+\dfrac{\sqrt{[r^2+c^2]}}{rc}$ is indeterminate for $c=0$. Its value is 0. Therefore $k_1=0$.

$$v=2r\sqrt{[\pi\rho]}\left[\frac{\sqrt{[r^2+c^2]}-r}{rc}\right]^{\frac{1}{2}}$$

where $c=r$. $v=2\sqrt{[\pi\rho r]}\sqrt{[\sqrt{(2)}-1]}$. But the attraction of the earth for a point at its surface is $\frac{4}{3}\pi\rho r=980$cm. per second.

$\therefore \pi\rho r=735$. $\therefore v=34.9$ cm. per second.

$v=\dfrac{dc}{dt}$. $\therefore \displaystyle\int\left[\frac{c}{\sqrt{[r^2+c^2]}-r}\right]^{\frac{1}{2}}dc=2\sqrt{[\pi\rho r]}t+k_2$, which, when integrated, gives t. $t=0$ when $c=0$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

377. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Expand in series $(1+x)^{1/x}$.

378. Proposed by C. N. SCHMALL, New York City.

Given $\sin^{-1}\dfrac{x}{a}+\sin^{-1}\dfrac{x}{b}+\sin^{-1}\dfrac{x}{c}=\pi$, solve for x.

379. Proposed by C. N. SCHMALL, New York City.

Given $y=ax+b$. If the values of x vary in an arithmetical progression, show that the value of y varies likewise.

GEOMETRY.

406. Proposed by DR. R. K. MORLEY, University of Illinois.

Given the lengths of a pair of conjugate diameters of an ellipse and the angle between them; to construct (with ruler and compass) the axes of the ellipse, *i. e.*, find their lengths and the angles they make with the given diameters.

407. Proposed by S. LEFSCHETZ, Ph. D., University of Nebraska.

To construct a right triangle knowing the sum of the sides of the right angle, and the sum of one of them plus the hypotenuse.

408. Proposed by ELMER SCHUYLER, Brooklyn, New York.

Given a point A on a circle and a chord of the circle; to draw a chord through A so that it shall be bisected by the given chord.

CALCULUS.

329. Proposed by C. N. SCHMALL, New York City.

Show that the general linear differential equation $\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\dots+P_{n-1}\frac{dy}{dx}+P_ny=V$, where P_1, P_2, P_3, ..., P_{n-1}, P_n, V are known functions of x (Edwards, *Integral Calculus*, p. 243), has a solution in the form $y=v_1\int v_2dx\int v_3dx\dots\int\frac{Vdx}{v_1v_2v_3\dots v_{n-1}v_n}$.

330. Proposed by C. N. SCHMALL, New York City.

X is a homogeneous function, in the nth degree, of x, y, z; Y is any function of u, v, w. If $ux=vy=wz=1+X\dots(1)$, prove that (by Euler's Theorem),

$$x\frac{\partial Y}{\partial x}+y\frac{\partial Y}{\partial y}+z\frac{\partial Y}{\partial z}+\frac{n}{x}\frac{\partial Y}{\partial u}+\frac{n}{y}\frac{\partial Y}{\partial v}+\frac{n}{z}\frac{\partial Y}{\partial w}$$
$$=(n-1)\left(u\frac{\partial Y}{\partial u}+v\frac{\partial Y}{\partial v}+w\frac{\partial Y}{\partial w}\right).$$

NOTES AND NEWS.

Editor Slaught attended the International Congress of Mathematicians at Cambridge, England. F.

We learn from *School Mathematics and Science* that Professor J. L. Gilpatrick of Denison University, died recently. Professor Gilpatrick was a subscriber of the MONTHLY from its beginning. F.

Dr. G. E. Wahlen, who has been editing the Department of Number Theory and Diophantine Analysis, will go to Europe on a year's leave of absence. Material for his department should, therefore, be sent to Editor Finkel.

The first part of volume 5, tome II, of the French mathematical encyclopedia appeared recently. This part contains 160 pages and is devoted to functional operations and functional equations, and to trigonometic interpolation. The last seven pages are devoted to the beginning of the article on spherical functions. According to a recent announcement in the *Jahresbericht der Deutschen Mathematiker-Vereinigung,* the first part of volume 2, tome IV, has also appeared. This part contains three articles entitled, respectively, Geometric foundations of statics, Geometry of masses, and Cinematics. A very large number of additional parts are announced as in press. Although more than twenty parts have been issued no volume has yet been completed. M.

Henri Poincaré, the most eminent French mathematician and physicist, died on July 17, 1912. He was born at Nancy, France, April 29, 1854, and he rose to fame as a young man in a meteoric manner. For a number of years he has been generally acknowledged as the most eminent mathematician, and numerous biographical sketches have appeared during recent years. The most complete extant sketch of his life work is probably the one published in 1909 by Ernest Lelon under the title "Biographie bibliographie analytique des ecrits," which contains about 80 pages, and was issued by the noted firm of Gauthier-Villars of Paris, France. In January of 1909, Poincaré was received as a member of the forty immortals, and, on this occasion, the director of the Académie Française, M. Masson, gave a comprehensive sketch of his life. A translation of this appeared in the September, 1909, number of *Popular Science Monthly.* A more recent sketch prepared by E. E. Slosson appeared in the October 5, 1911, number of *The Independent* under the title "Twelve major prophets of to-day — III." M.

At the meeting of the National Education Association in Chicago, the round table conference of the Secondary Section was held on Wednesday, July 10, at 9 o'clock. The chairman is Charles M. Austin of the Central High School, Minneapolis. The amended report of the Committee of Fifteen on Geometry Syllabus was presented by Professor E. R. Hedrick of the University of Missouri, who is chairman of the sub-committee on the syllabus proper. Since the report was presented in San Francisco last July full and careful consideration has been given to all criticisms and suggestions and in the light of these numerous modifications and additions have been made, and it is in this completed form that the report will now be given by Professor Hedrick. This revised form of the report has been printed and is in process of distribution through the Department of Education in Washington. Word has just been received that the edition of 5000 copies is nearly exhausted, which would indicate that another edition will need to be issued after the Chicago report has been made. The historical introduction was included in the proceedings of the San Francisco meeting. S.

ERRATA.

Page 27, line 5, Lefschetz's article, for "Caily" read Cayley.

Page 34, counting from top of page, omit line 6.

Page 59, last line, for + read −.

Page 82, problem 376, for "a_n—k" read a_{n-k}; and in last line, for "a_n" read a_8.

Page 102, third paragraph, third line, for "$\angle BOP$" read $\angle AOP$.

Same paragraph, fifth line, for "P'_1P_1B" read P'_1P_1A.

Page 105, fifth line, for "P'_1P_1B" read P'_1P_1A.

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A LABOR-SAVING DEVICE FOR SERIAL MULTIPLICATION OR DIVISION BY MEANS OF AN ARITHMOMETER IN CASES WITH SMALL DIFFERENCES OF CONSECUTIVE RESULTS.*

By E. R. MARSHALL, Ph. D., New York City.

INTRODUCTION.

In computing the present value of future expense charges on annual dividend policies for a life insurance company in New York, I had to multiply a constant for every five years of age, representing a certain percentage of the mean of the office premium and the net premium for ordinary life, by a decreasing annuity for the different policy ages. The results obtained were taken only to the nearest integer. As there happened to be repetitions of some integers and omissions of others with what seemed to be absolute irregularity, I thought of trying to determine the limits between the factors giving one integer and those giving the next. This led me to devise a method of serial multiplication by means of the arithmometer (later extended also to serial division) for such and similar kinds of work, which eliminates most of the labor required in the ordinary methods if the differences of the consecutive results do not exceed a few units,—and a large proportion of it for differences greater than a few units. The present paper is an explanation of this device, which, I believe, may prove of interest also to other actuarial computers, as well as to mathematicians in general.

SERIAL MULTIPLICATION.

I shall first treat of serial multiplication when used for finding the nearest or the highest integers of a series of consecutive products with a constant factor, whose differences do not exceed a few units—the actual case that gave rise to the discovery of the device.

*Presented to the American Mathematical Society at its April meeting.

Let $x_{0,1},\ x_{0,2},\ \ldots,\ x_{0,\,k},\ \ldots;\ x_{1,1},\ x_{1,2},\ \ldots,\ x_{1,\,k},\ \ldots;\ \ldots;$
$x_{i,1},\ x_{i,2},\ \ldots,\ x_{i,k},\ \ldots;\ \ldots$

represent the required products, in which the first subscripts indicate their nearest or their highest integers, as the case may be; and let

$$a_{0,1},\ a_{0,2},\ \ldots,\ a_{0,\,k},\ \ldots;\ \ldots;\ a_{i,1},\ a_{i,2},\ \ldots,\ a_{i,k},\ \ldots;\ \ldots$$

be the corresponding given values of the variable factor, while m is the constant factor. So that we have,

$$x_{0,1}=ma_{0,1},\ \ldots,\ x_{0,\,k}=ma_{0,\,k},\ \ldots;\ \ldots;\ x_{i,1}=ma_{i,1},\ \ldots,\ \ldots$$

and, in general, $x_{i.k}=ma_{i,k},\ \ldots$

Hence

$$a_{i,k}=\frac{1}{m}x_{i,k} \qquad (1).$$

Also,

$$x_{0,\,k}<.5\ (\text{or }1)\leqq x_{1,\,k}<1.5\ (\text{or }2)\leqq x_{2,\,k}<2.5\ (\text{or }3) \qquad (2)$$

— the double symbol $\leqq$ having reference to the value of k*$=1$ only, and reducing to the single symbol $<$ for all other values of k.

Multiplying (2) by $1/m$ and substituting from (1), we obtain

$$a_{0,k}=\frac{1}{m}x_{0,k}<.5\times\frac{1}{m}\ (\text{or }1\times\frac{1}{m})\leqq\frac{1}{m}x_{1,k}=a_{1,k}<1.5\times\frac{1}{m}\ (\text{or }2\times\frac{1}{m})$$
$$\leqq\frac{1}{m}x_{2,k}=a_{2,k}\ \ldots\ \text{etc.},\ \ldots(3),$$

the double and the single symbols having the same reference as before; or simply,

$$a_{0,k}<.5\times\frac{1}{m}\ (\text{or }1\times\frac{1}{m})\leqq a_{1,k}<1.5\times\frac{1}{m}\ (\text{or }2\times\frac{1}{m})\leqq a_{2,k},\ \ldots,\ \text{etc.};$$

and, in general,

$$a_{i,k}<\frac{1}{m}(i+.5)\leqq a_{i+1,k}<\frac{1}{m}(i+1+.5) \qquad (4a);$$

*In this notation, k merely marks a certain number of the set with which we are dealing, and it is convenient to use $k=1$ to mark the first number considered, whose nearest or highest integer is i.

for the case of the nearest integers, or

$$a_{i,k} < \frac{1}{m}(i+1) \leqq a_{i+1,k} < \frac{1}{m}(i+2) \tag{4b}$$

for the case of the highest integers,— for all values of i from 0 up (where the the sign of equality of the double symbol *may* hold only for $k=1$).

Hence, $\frac{1}{m}(i+.5)$, or $\frac{1}{m}(i+1)$, are the critical values for locating all the lower values of $a_{i,k}$, which produce an integer i *at most*, and all the higher values of $a_{i,k}$, which produce an integer $i+1$ *or greater*. Moreover, all the values of $a_{i+1,k}$, giving the integer $i+1$ (for $i=0, 1, 2, \ldots$, etc.), lie between $\frac{1}{m}(i+.5)$ and $\frac{1}{m}(i+1.5)$ for the nearest integer case, or between $\frac{1}{m}(i+1)$ and $\frac{1}{m}(i+2)$ for the highest integer case,— with the very rare possibility of *one* of these values equaling the lower limit for $k=1$.

What we have to do, therefore, to find the a's (if any) producing a given integer $i+1$, is to multiply a sufficient approximation of $1/m$ successively by $i+.5$ and $i+1.5$, in the one case, and by $i+1$ and $i+2$, in the other, and assign the product $i+1$ to all the a's between the obtained products. There may be several such a's, in which case we save the labor of multiplying each of them by m in the direct process; and there may be no a whose value lies between the obtained limits, thus showing that there could be obtained no integer $i+1$ in the direct process of multiplying the a's by m. In the latter case we add the value of $1/m$ once, twice, etc., times to the higher of the limits last obtained, until we get the first of the products, $\frac{1}{m}(i+1.5+h)$ or $\frac{1}{m}(i+2+h)$, just exceeding some $a_{i+1+h,1}$, or (very rarely) just equal to some $a_{i+2+h,1}$. Then all the a's immediately below this product and above the limit previously obtained, *i. e.*, $a_{i+1+h,1}$, $a_{i+1+h,2}$, $\ldots$, $a_{i+1+h,l}$, give the integer $i+1+h$, while $a_{i+2+h,1}$ gives the integer $i+2+h$ in the very rare case of exact equality. In either of the last mentioned two possibilities, $i+1+h$ or $i+2+h$, is the first integer above i that could be obtained also in the direct process of multiplication.

EXAMPLES.

The process is best illustrated by one or two concrete examples:

Let $m=5.135$ and let the variable factors in their order of magnitude be those given in the following table, which for convenience is broken up into six partial columns:

1	2	3	4	5	6
.131	.290	.506	.902	1.617	2.789
.158	.326	.569	1.017	1.815	3.078
.189	.365	.637	1.143	2.031	3.383
.221	.403	.710	1.281	2.266	3.705
.255	.451	.800	1.439	2.519	

Let us further suppose that the computer is seeking the nearest integers of the corresponding products* and is using a "Saxonia" arithmometer for the purpose. The actual products of the whole of his work lying within a range less than 100, it is sufficient to take $1/m=.19574$, exact to the fifth decimal place, since $a_{i,k}=\frac{1}{m}\times x_{i,k}<\frac{1}{m}\times 100$, which will make $a_{i,k}$ correct to the nearest third decimal place, as given in the table of values of a's. Now, putting .19474 on the fixed plate of the "Saxonia," and multiplying it by .5, we obtain .097370, and taking only the needed three decimal places .097, we find it less than the smallest $a=.131$. Therefore, we get no values zero for any of the nearest integers sought. Next we move over the slide one station to the right, efface the 5 in the first "quotient hole," but leave the original product on the slide, and turning the handle once, we get .292110, or $.292>.290$ and all the five given a's below it. Therefore, we write 1, the number in the second quotient hole from the right, opposite every one of the six lowest a's. We throw up the multiplicand once more on the slide, and we get .48685, or $.487>.451>.403>.365>.326$. Hence we write 2 opposite these four numbers. Similarly, we get by another turn of the motive handle, 3 opposite the next higher three numbers, and by still another turn 4 opposite the next two and so on until for nine turns we obtain the nearest product $1.850>1.815$, giving nine for this factor. We turn the handle again, and we get 8, colored red,* in the quotient hole. This gives the computer warning that the 8 in the second quotient hole really corresponds to the tenth turn of the handle additively. The nearest number obtained in the product is 2.045, giving 10 opposite the given $a_{10,1}=2.031$. The operator could go on turning the handle and obtaining red 7, 6, 5, etc., in the reversed order, and consider them equivalent to black 11, 12, 13, etc.—for which we have the rule that each of the first red figures, f_r, turned up additively immediately after the first nine black figures, is equivalent to $f_r+2(9-f_r)$ given in black; thus, $5_r=5+2(9-5)=13$, etc. But it is better, as soon as the operator receives the warning of the first red figure, to efface it in its quotient hole, and put down the equivalent number in black color; after which he may go on turning the handle addi-

* In the "Saxonia" arithmometer the figures showing how many times a number has been taken subtractively are distinguished by the red color from those showing how many times a number is taken additively, which are given in black.

tively, obtaining black numbers again until we reach 19, followed again, in the next turn, by 1 black in the tens and 8 red in the units, which is equivalent to twenty turns. Here once more the rule would hold that a number with red units is equivalent to the same number plus twice the excess of 19 over it; or in symbols, $n_m = n_b + 2(19 - n_b)$, where n_m is a number in mixed color immediately after the first nineteen additive turns, and n_b is the same number in purely black color, *e. g.*, $17_m = 17_b + 2(19 - 17_b) = 21_b$.

In our case the computer will find that the nearest product corresponding to ten turns is 2.045, giving 10 as the nearest integer derived from the factor 2.031. The nearest product corresponding to eleven turns is $2.240 < 2.266$. Hence, between $\frac{1}{m} \times 10.5 = 2.045$ and $\frac{1}{m} \times 11.5 = 2.240$ there is no factor, the two consecutive given factors being 2.031 and 2.266, one lower than the lower limit, and the other higher than the higher limit. We therefore turn the handle once more, finding the next nearest product 2.434 > 2.266, giving 12 for this factor. If our table should also contain the factors 2.320, 2.390, and exactly 2.434, the integer 12 would correspond also to the first two additional factors, but the last factor would give 13 as the nearest integer in the direct process, and, as has been proven above, must also be assigned 13 as its nearest product in our work.

On the other hand, if our table of given factors would miss the factors 1.617, 1.815, 2.031, 2.266, the factors, namely, corresponding to the integers 8, 9, 10, 12, then, after having thrown up on the slide 1.46055, giving the integer 7 for the factor 1.439, we should have to turn the handle continuously six times more to get the nearest product 2.629, the first to exceed the given factor 2.519, which would, therefore, have 13 for its nearest product. Now, in such a case, while turning the handle and keeping our eye open upon the products thrown up until we would notice one greater than 2.519, the factor next higher than 1.439, we would naturally be likely to overlook the last 9 in black color in the "quotient holes" or even perhaps the first red figures 8, 7, 6, until we came to red 5. The rule, therefore, that $5_r = 5_b + 2(9 - 5_b) = 13_b$ is of advantage for such a case, as it enables us to relieve our attention for a while from the "quotient holes," and fix it wholly upon comparing the thrown up consecutive products with the column of given factors.

We see from our example that, besides the time spent in obtaining $1/m$ to five decimal places and in setting upon the machine the quotient .19474 and making the five preliminary turns to insure correctness in the third decimal place, we have only to make nineteen turns of the handle and, in addition, to replace a red figure by its equivalent black number, in order to enable us to write down all the required integers corresponding to the given twenty-nine factors, or to a similar series of factors of which the last one is less than 3.79743. Whereas in the direct process of multiplication

we would need, after setting up 5.135, the constant factor, to multiply it by the twenty-nine given co-factors, each containing from three to four figures, or an average of 101 figures (by actual count in this example 100 figures), each figure requiring, on an average, $\frac{0+9}{2}=4\frac{1}{2}$ turns, which makes in the aggregate about 455* turns of the motive handlé, besides the twenty-nine effacings of the products, in order to find the same twentynine integers. Of course, some time might be gained also in the direct process of multiplication, by forming the differences of the factors; but not very much, since these differences are not always small, growing up in our example to .322 (=.705—3.383), besides involving the risk of carrying any error in the middle of the process to the very end. We see, therefore, that, discounting the time spent in registering the integers, which is in the two processes the same, our indirect process, which may be called *the index process* (as the proper first indexes of the different $a_{i,k}$'s are directly thrown up on the slide in the "quotient holes") would be *at least* about twenty-four times shorter than the direct process, namely, $\frac{455}{19}=24$, even neglecting the twenty-nine effacings, and about ten or twelve times shorter than by the process of differences.† Moreover, after a little practice with the index method one is enabled with great ease to turn the handle with the left hand and do the registering of the obtained integers beside the given factors with the right hand, in the case where each factor has another integer belonging to it, and where some integers may be entirely missing, where, consequently, we have to turn the handle at least once and, frequently, even twice, three, or four times, before we arrive at the integer belonging to a new factor. The kind of work where this occurs is just as frequent in practice as the kind of work represented by our example, where one integer belongs to several factors.

In all, I think, it is safe to assume that the index process would save at least seventy-five to ninety-five per cent of the work (excluding the registering); and more frequently the higher percentage of the work is saved. Besides, practice will show that it is also a very safe and reliable process, much less subject to error than either the direct or the difference process.

In our example we have supposed that we were looking for the nearest integers. In case the highest integers are sought, the process will be in all respects identical, except that, instead of multiplying originally $1/m$ by .5, effacing the 5 in the first "quotient hole" of the slide, and moving the slide over one station to the right, in which the indexes 1, 2, etc., are turned up, we have to start with multiplying $1/m$ by 1, and, effacing it, proceed directly to turn up the required indexes in the first "quotient hole."

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* In the actual example only 373 turns are needed, as the co-factors happen to abound in 0's, 1's and 2's. Other similar series of co-factors might happen to abound in 7's, 8's, and 9's, increasing the number of turns in the direct process, without, however, affecting the differences of consecutive results.

† It is hardly necessary to explain why in the fraction $\frac{455}{19}$, we must neglect the preliminary five turns, as these would also be sufficient even for a series of one hundred factors instead of twenty-nine, besides neglecting all the effacings in the direct process, involving a much greater waste of time.

Serial Division.

Looked at from another point of view, the index process just explained gives us a short method of obtaining the nearest or highest integers of $x_{i,k}=a_{i,k}\div\frac{1}{m}$, or the nearest or highest integers of the quotients obtained in dividing the different a's by $1/m$. It will, therefore, easily be seen that this method can be used with great advantage also for performing a series of divisions in which the divisor is a constant and the consecutive quotients differ from each other by a few units. Then we treat the divisor as we have treated $1/m$ in multiplication, putting it on the board of the arithmometer to the right, and proceeding to multiply it in exactly the same manner as in the case of multiplication, until we get on the slide the first number above the lowest dividend, and then the first number above the next higher dividend, etc. The corresponding multipliers appearing in the quotient holes will be the required quotient integers. Obviously in this case the method might more properly be called the "checking method of division," as the proceeding is virtually the same as would be followed in checking the original quotients by multiplying the constant divisor by each of them and comparing the results with the corresponding dividends.

In trying to extend the field of usefulness of this method by applying it to divisions with larger differences, it will be found that when these latter are too large (exceeding three or four figures), its use as an independent initial method would be of doubtful practical utility, as too many precautions would be requisite to insure accurate results. Yet for *checking* work of this character done by the direct method, it can be used with great speed and expedition, as no special precautions are necessary in this case. and the labor of putting up each new dividend on the sliding plate, as well as of effacing the quotients, would be wholly saved.

Extended Application of the Method.

There is, however, a large field of actuarial work, comprising divisions with but moderately large differences, not exceeding two or three figures, where the application of our "checking method" of serial division used as an independent, initial method, would prove of decided advantage. The actuarial field referred to consists of the kind of computations represented by Mr. George King's conversion and valuation tables appended to his paper "On Policies with Deferred Participation in Profits, and Policies with contingent Bonuses," read before the Faculty of Actuaries in Scotland, and printed in the Transactions of the Faculty, Vol. V, Part IX, No. 53, 1911. The peculiar feature of this work, computed to the third decimal place, is the comparatively small differences of the consecutive results, which may be obtained as the quotients of a number of separate series of divisions, each with the same divisor. Table II, *q. v.*, may serve as an in-

stance, giving the factors for converting immediate cash bonuses into endowments maturing at the end of the deferred bonus term. The individual members of this table are the reciprocals of pure endowments, ${}_tE_x^{-1}=D_x \div D_{x+t}$, where x is the age attained and t is the remaining bonus term, D_{x+t} being constant for each series of divisions.

Mr. King recommends the method of reciprocals and differences as the best and speediest for the computation of these tables, and this would, of course, imply the series with D_x constant for each, instead of D_{x+t}. This method is undoubtedly the best of all known up to date. A much greater saving of labor might, however, be effected through the application of our "checking method of serial division," with one or two simple precautions. In the first place the computation of the reciprocals of the divisors, which for different tables are different, including such compound expressions as $M_{x+t}-M_x+D_z$, would be wholly dispensed with, as well as the finding of their differences. In the second place, the differences of the consecutive results, which in our method are the consecutive multipliers, would be found to be considerably smaller than those of the reciprocals of the divisors; thus, while the former mostly range below 100, and rise to a few hundred only at high ages combined with long periods, which is hardly normal, the latter range mostly several hundred, and rise even above several thousand at corresponding high ages and periods, if calculated to insure results correct to the third decimal place; there would, consequently, result a further considerable reduction of labor in the process of arithmometer multiplication.*

An Illustration of the Method.

It will perhaps not be considered superfluous to state in detail the peculiar features of the procedure in this case, where the differences of the results consist of two or three figures:

Placing the divisor, in our example D_{x+t}, on the arithmometer board to the right and multiplying it by .5 of .001 to insure the correctness of the results to three decimal places, we move over the sliding plate four stations to the right, since $D_x \div D_{x+t}$ will give for normal values of x and t an improper fraction less than 10, which, consequently, has, besides

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$$*(D_x-D_{x+1}) \div D_{x+t}=[(D_x-D_{x+1}) \div D_x] \times \frac{D_x}{D_{x+t}}=[1-a_{x\,\overline{1|}}]\dagger \times \frac{D_x}{D_{x+t}},$$

taken correct to the third decimal place, is the formula for the differences of consecutive results giving mostly *two significant figures*, and, at most, three for high ages and long periods; and

$$\frac{1}{D_{x+t+1}}-\frac{1}{D_{x+t}}=\frac{D_{x+t}-D_{x+t+1}}{D_{x+t} \times D_{x+t+1}}=\frac{1-a_{x+t\,\overline{1|}}}{D_{x+t+1}}$$

taken correct to the eighth decimal place, is the formula for the differences of consecutive reciprocals, giving three significant figures so long as D_{x+t+1} consists of five figures and $1+a_{x+t\,\overline{1|}}$ remains below 1, and four significant figures when D_{x+t+1} is reduced to four figures only in the integral part, while $1-a_{x+t\,\overline{1|}}$ approaches the value of .1, that is, when $x+t+1$ becomes equal to 70 or thereabout, and around 80 the number of significant figures becomes five. Of course, the O^m Table at three per cent of the British Offices Life Tables, 1893, is used by me for for illustration,—this being the same used by Mr. King in his computation.

† $\overline{1|}$ is used for the symbol $\overline{1}|$. Ed. F.

the three decimal places required, one figure in the integral part, or four figures in all. We then multiply the divisor from left to right by a number just sufficient to make the product exceed the lowest D_x, corresponding to the highest age of the table, which will be the lowest quotient of the series for the given divisor. To obtain this number we turn up in the first to the left quotient hole sufficient units to make the product just exceed the given dividend D_x, and turning then the handle subtractively once, we move over the sliding plate one station to the left, and here again we turn up sufficient units, of the next lower order, to make the product just above the given dividend, turning then again the handle backwards once, and so on, until we obtain the figure in the fourth from the left quotient hole which will just make the product exceed our dividend. This will be the proper figure of the third decimal place. After this we work only the third and second quotient holes counting from right to left, corresponding to the second and third decimal places of the difference between the result previously obtained and the result corresponding to the next higher dividend D_{x-1}. These two figures and each of the subsequent sets of two figures are obtained in the same manner as the original four figures, by working from left to right and correcting each figure of the higher order (of the second decimal place) by turning the handle subtractively once so as to obtain a product just lower than the corresponding dividend. The only precautions requisite to insure correct results are: first, that we work the motive handle only additively, except when correcting any of the figures of the higher orders by turning subtractively once, so as to pass from a figure giving a product just above a given dividend to one giving a product just below it; second, that we correct an occasional red-figured number n_r by a corresponding black-figured number according to the rule, n_r = arithmetical complement of (n_r+2) to the next higher unit+one such unit.*

A Concrete Example.

A concrete example will show that the method in practice works out even easier than in theory:

Starting with $D_{46} \div D_{46}$, for our Table II, *i. e.*, with the divisor $D_{x+t} = D_{46}$ and $t=0$, we multiply $D_{46}=20622$ by $.0005+1$, and efface 5 in the extreme right hand quotient hole, knowing beforehand that the result in this case is 1.000. The product obtained is 20632.311, a number just above the dividend D_{46}. We might, if we wished, verify the result by subtracting .001 times the divisor, and we would get 20611.689,—a number just below our dividend, proving that .999 would be .001 below the true result.

* This rule is a slightly modified and more generalized form of the corresponding rule given above, and its proof is easy. Since after the black figure 9 come the red figures 8, 7, etc., in the reverse to the natural order, it is evident that $x_r=9+(9-x_r)=10+[10-(x_r+2)]$, where x_r is one red figure. Now, assuming, for instance, that our red-figured number, N_r, consists of three figures, we have, $N_r=100z_r+10y_r+x_r=100[10+10-(z+2)]+10[10+10-(y+2)]+10+10-(x+2)=1000+1000-100z+200-200-10y-20+20-x-2=1000+1000-(100z+10y+x+2) = 1000+[1000-(N_r+2)]$.

Moving over the slide three stations from its extreme left position to the right, we find that five additional turns of the handle give 21663.411 > 21489 $=D_{45}$. We therefore turn the handle subtractively once and obtain 21457.191 $<D_{45}$, and moving over the slide one station to the left, we must turn the handle twice before obtaining the product 21498.435, just above D_{45}. We have then in our quotient holes the number 1.042, which is the required quotient corresponding to $D_{45} \div D_{46}$.

Assuming now provisionally the next difference to be again .042, we find, after turning the handle four and two additional times in the proper quotient holes, that the product obtained is 22364.559 $<D_{44}=$ 22379. We therefore turn the handle once more for the third decimal place, and obtain 22385.181, a number slightly above D_{44}. The corresponding quotient in the quotient holes is then found to be 1.085. Assuming again the next difference to be, like the last one, .043 or thereabout, we turn the handle the indicated number of times for each of the two corresponding quotient holes, finding the product corresponding to the difference .044 = 23292.549 $<D_{43}=$ 23295. We, therefore, turn the handle once more for the third decimal place, and obtain the product 23313.171, slightly above D_{43}, giving in the quotient holes 1.0[68], the number in brackets given in red color. By the above rule for converting a red-figured number into a black-figured one, this is equivalent to 1.100+arithmetical complement of 70=1.130, which is therefore the true quotient corresponding to $D_{46} \div D_{43}$. It should be observed that we took the difference .042 and .043 as a guide in each of the subsequent cases only to shorten slightly the process of multiplication; but it was not absolutely necessary; we could have found the correct subsequent figures by turning the handle each time for the coroesponding second decimal place *tentatively* five times, and by turning the handle once subtractively on finding the corresponding product too high, as done above for obtaining the quotient $D_{45} \div D_{46}$, and as explained in the theoretical statement. Similarly, to obtain the first quotient of the series under consideration, we have availed ourselves of the fact that the dividend was in this case equal to the divisor. If this is not the case, a more literal following of the directions given in the theoretical statement for the procedure would entail but a very little extra labor in finding each of the figures of the higher orders, beginning from the left, by a tentative number of additive turns of the handle producing a product just above the corresponding dividend, followed by one subtractive turn of the handle to reduce it again to a number just below the dividend. This procedure ought now to be perfectly clear from the illustration.

It should further be noticed that there is no red zero in the "Saxonia" arithmometer, but a black zero may sometimes mean the equivalent of a red zero— namely, when the former in a quotient hole is derived from 9 by adding 9. We should, therefore, be careful to distinguish between a black zero proper and one derived in the above manner, which is equivalent to 18 also by the

above rule for conversion of red-figured numbers into black-figured ones.

CONCLUSION.

In conclusion I wish to say that the application of the "checking method" to serial division would also prove of considerable advantage for computing temporary and deferred annuities by means of the arithmometer. The usual continued method of arithmometer computation of these functions is based upon the formulas: $|_{n+1}a_x = |_n a_x + D_x^{-1} \times D_{x+n+1}$, and $_{n+1}| a_x = _n| a_x - D_x^{-1} \times D_{x+n+1}$, respectively,—making D_{x+n+1} the out-factor, which is quite large. If, however, we take as the basis of our computation the corresponding formulas: $|_n a_x = \frac{N_x - N_{x+n}}{D_x}$ and $_n| a_x = \frac{N_{x+n}}{D_x}$, then, beginning for any given entrance age x with the smallest dividend $N_x - N_{x+1} = D_{x+1}$ for $n=1$ and proceeding downwards to the largest dividend $N_x - N_{w-1}$, for temporary annuities, and from N_{w-1} upwards to the largest dividend N_x for deferred annuities (w being the highest age of the mortality table), we shall evidently save much labor in the continued multiplication of D_x required by our method, in which the out-factors are the differences of the required annuities themselves, since these differences begin with three significant figures and end with zero for temporary annuities, and conversely for deferred annuities, whereas the D_x's consist mostly of five significant figures. Moreover, as these differences change but gradually, we can also save labor by using provisionally for each additional multiplier a number as near the preceding multiplier as *convenient* and then correcting it so as to obtain an aggregate product just above the corresponding dividend. In practice we may also make use of such device as multiplying by the next higher unit of a given figure and subtracting the arithmetical complement of the figure times the multiplicand, provided that this arithmetical complement is smaller than the figure in the same quotient hole previously recorded, so as not to come to any *red figure* in the quotient *by subtraction.* For instance, after having found $|_1 a_{40}$ at 3% to be .962, while $|_0 a_{40} = 0$, we assume the next difference also to be approximately .962, and multiply the divisor D_{40} first by 1. and then by $-.1$, producing the aggregate multiplier (quotient) 1.862, which, however, gives a product less than N_{41}, and we must add to our multiplier .025, bringing up the aggregate multiplier to 1.887, the true value of $|_2 a_{40}$, in order to obtain an aggregateproduct just above N_{41}. We thus save the labor of multiplying the divisor originally by .9 and then changing the obtained red figure 9 to the black-figured number 1.8 by the conversion rule given above.

A still greater amount of labor can be saved if the annuities are computed only for valuation purposes where there is neither possibility nor need for great accuracy. In such a case it is much better to take as our basis for the computation the formulas: $|_n a_x = {}_1E_x + {}_2E_x + \ldots + {}_nE_x$, and $_n| a_x =$

${}_{n+1}E_x + {}_{n+2}E_x + \ldots$ etc.,—a sum of pure endowments in either case. The pure endowments, ${}_tE_x = D_{x+t} \div D_x$, are first computed to three decimal places by our "checking method" as explained above, and then these are added successively for the same value of the entrance age x, from $D_{x+1} \div D_x$ to $D_{x+n} \div D_x$ for temporary annuities, and from $D_{w-1} \div D_x$ to $D_{x+n+1} \div D_x$ for deferred annuities. Of course, in the successive addition of these endowments, themselves correct only to the third decimal place, there will of necessity arise a small accumulation of error as we proceed further from the starting point, but the maximum aggregate error cannot amount to more than a few units in the third decimal place, which for purposes of valuation is rather insignificant. The preliminary endowment tables themselves should be computed by starting for each entrance age x from the lowest dividend D_w and going upwards to the dividend D_x, by adding continually to the variable multiplier (quotient) until we reach the highest quotient 1.000. This procedure would prove a great labor-saver even in comparison with our own more accurate method of annuity computation previously explained, based upon the formulas: $| {}_n a_x = [N_x - N_{x+n}] \div D_x$ and ${}_n | a_x = N_{x+n} \div D_x$, since the differences of the consecutive pure endowments are considerably smaller than the corresponding differences of the annuities themselves. This comparison in favor of the pure endowment basis of computation is especially true in the case of *decreasing* or *increasing* temporary and deferred annuities, in which the summations $\sum_{i=1}^{i=n} r_i D_{x+i}$ replace the N_x's in the numerators of the above formulas. In such a case, the preliminary computations of the Σ's, and of their differences for temporary annuities, would be wholly dispensed with, the much simpler summations of the pure endowments taking their place at the very end of the procedure.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

371. **Proposed by ELMER SCHUYLER, Brooklyn, N. Y.**

In a G. P. of an odd number of terms, all of the terms being positive, and the ratio different from 1, show that the middle term is less than the arithmetical mean.

I. **Solution by ELIJAH SWIFT, Princeton University.**

We have a G. P. of an odd number of positive terms, a, ar, ar^2, ..., ar^{2n}.

To prove $ar^n < \dfrac{a(r^{2n+1}-1)}{(2n+1)(r-1)}$.

Assuming $r>1$ and clearing, we have to prove, a being positive,

$$(2n+1)(r^{n+1}-r^n) < r^{2n+1}-1; \text{ or}$$
$$r^{2n+1}-(2n+1)r^{n+1}+(2n+1)r^n-1>0).$$

Call the expansion on the left $f(r)$. Then $f(1)=0$.

$$f'(r)=(2n+1)r^{n-1}\{r^{n+1}-(n+1)r+n\}.$$

Call the expression in brackets $\phi(r)$. Then $\phi(1)=0$.

$$\phi'(r)=(n+1)\{r^n-1\}.$$

Then we have, since $r>1$, $n>0$, $\phi'(r)>0$.

Since $\phi(1)=0$ and $\phi'(r)>0$, $\phi(r)>0$. But $f'(r)=(2n+1)r^{n-1}.\phi(r)$. Therefore $f'(r)>0$, and finally, $f(r)>0$.

The case where $r<1$ may be treated by reversing the inequality signs.

Also solved by H. Prime, H. C. Feemster, C. E. Githens, A. M. Harding, and the Proposer.

372. **Proposed by S. LEFSCHETZ, Ph. D., University of Nebraska.**

Prove that $\sum\limits_{n=1}^{n=\infty} n^2x^{n-1} = \dfrac{1+x}{(1-x)^3}$, if mod. $x<1$. (*Schlömilch.*)

Solution by H. C. FEEMSTER, York College, York, Nebraska.

$$\sum_{n=1}^{n=\infty} n^2x^{n-1}=1+4x+9x^2+16x^3+\ldots+r^2x^{r-1}+(r+1)^2x^r+(r+2)^2x^{r+1}+\ldots.$$

and $(1-x)^3\sum_{n=1}^{n=\infty} n^2x^{n-1}=[1+4x+9x^2+16x^3+\ldots+r^2x^{r-1}+(r+1)^2x^r$
$+(r+2)^2x^{r+1}+\ldots]-[3x+12x^2+27x^3+\ldots+3(r-1)^2x^{r-1}+3r^2x^r$
$+3(r+1)^2x^{r+1}+\ldots+[3x^2+12x^3+\ldots+3(r-2)^2x^{r-1}+3(r-1)^2x^r$
$+3r^2x^{r+1}+\ldots-[x^3+\ldots+(r-3)^2x^{r-1}+(r-2)^2x^r+(r-1)^2x^{r+1}+\ldots]$
$=1+x$, so

$$\therefore \sum_{n=1}^{n=\infty} n^2x^{n-1}=\frac{1+x}{(1-x)^3}, \text{ as required.}$$

Also solved by H. Prime, J. Scheffer, A. M. Harding, and Elijah Swift.

373. Proposed by X, National Electric Light Association, Brooklyn, N. Y.

(a) For underground distribution of direct current electrical energy, we have $DA^n-CB^n=H$, where the only unknown is n, which represents the number of years it will take a large direct current low tension feeder to pay by line loss saving for the increased investment over a smaller feeder.

(b) $bVf^2l^2\beta^2+aVf\beta^{1.6}=W$, the iron loss equation which is to be solved for β. When $bVf^2l^2\beta^2$ represents the eddy current loss in the core of a transformer, and $aVf\beta^{1.6}$ is the hysteresis loss in the core. a, b, and l are constants of the core, V is the voltage, f is the frequency, and β is the flux density and is the only unknown in the equation.

Letting $bVf^2l^2=A$, and $aVf=C$, we have $A\beta^2+C\beta^{1.6}=W$.

Solution by E. B. ESCOTT, Ann Arbor, Michigan.

(a) $DA^n-CB^n=H$, n unknown. Since $A=10^{\log A}$ and $B=10^{\log B}$, the equation may be written

$$D.10^{n\log A}-C.10^{n\log B}=H.$$

Putting $10^n=x$, the equation becomes $Dx^{\log A}-Cx^{\log B}=H$.

This equation and the one under

(b) $$A\beta^2+C\beta^{1.6}=W$$

are trinomial equations, which may be solved in various ways. One of the simplest methods to use in practice is by addition and subtraction of logarithms. This is given in C. Runge, *Praxis der Gleichungen* (Leipzig, 1900), pages 140-157.

Gundelfinger has given tables to three decimals which enable one to

solve trinomial equations of all degrees with any coefficients. S. Gundelfinger, *Tafeln zur Berechnung der reellen Wurzeln sämtlicher trinomischer Gleichungen* (Leipzig, 1897).

For references to the literature of trinomial equations, see the article, R. Mehmke, *Calculs numériques*, in the *Encyclopédie des Sciences Mathématiques*, tome I, Volume 4, I 23.

In § 41, pages 320-325, the following method is given.

In $A\beta^2+C\beta^{1.6}=W$, put $\beta=kx^n$. Then $Ak^2x^{2n}+Ck^{1.6}x^{1.6n}=W$.

Let $1.6n=1$, whence $n=\dfrac{1}{1.6}=.625$.

Let $Ak^2=Ck^{1.6}$, whence $Ak^{0.4}=C$, and $k=\left(\dfrac{C}{A}\right)^{2.5}$.

Then equation reduces to $x^{1.25}+x=D$.

Make a table of values of $x^{1.25}+x$ for different values of x with as small an interval as desired. Then any given equation may be solved by simple interpolation.

For graphic solution (by nomography), see Articles 45, 51, 61; also M. d'Ocagne, *Traité de Nomographie* (Paris, 1899), pages 367, 387.

Another method is given by F. Schlepps, *Euber die Auflösung trinomischer Gleichungen aller Grade* (Halle a. S. (1899), 15 pages). [See *Jahrbuch ü. d. Fortschritté der Math.*, Vol. 30, page 104.]

GEOMETRY.

400. Proposed by FRANCIS RUST, C. E., Pittsburgh, Pennsylvania.

Given a circle and a point P without; construct, *using the straight edge only*, the two tangents to the circle through P.

Solution by H. E. TREFETHEN, Colby College, Waterville, Maine; H. C. FEEMSTER, York College, York, Nebraska, and ELMER SCHUYLER, New York City.

From P draw two secants and complete the inscribed quadrilateral thus determined. Let PQ be the external diagonal and R the point of intersection of the internal diagonals. Hence each side of the triangle PQR is the polar of the opposite vertex and QR cuts the circle in S and T, the points of contact of the required tangents PS and PT. Thus the construction is effected with the ruler only.

Also solved by M. E. Graber and E. B. Escott.

401. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Find by Euclidean geometry a point whose distances from the vertices of an equilateral triangle are in the ratio 3:4:5. The general case of ratio $a{:}b{:}c$ would prove interesting.

I. Solution by W. J. GREENSTREET, M. A., Editor The Mathematical Gazette, Burgfield, England.

Let ABC be any triangle. The locus of a point P such that $PA{:}PB=x{:}y$ is well known to be a circle. Let the circular loci of the points P given

by $PA:PB=x:y$, $PB:PC=y:z$ cut in P. Then since $PC:PA=z:x$, it follows that P lies on the third of three circular loci given by $PA:PB:PC=x:y:z$, and these are three coaxial circles.

Each is orthogonal to the circle ABC. Hence the two points of intersection of the three loci are inverse points with respect to the circle ABC.

Hence two points satisfying the conditions can be found.

It follows that when P is on the circle ABC the two solutions reduce to one. Apply Ptolemy's theorem and we have

$$a'x+b'y=c'z, \quad b'y+c'z=a'x, \quad \text{or} \quad c'z+a'x=b'y,$$

where a', b', c' are proportional to the sides of the triangle ABC. Hence, if ABC is equilateral, we have

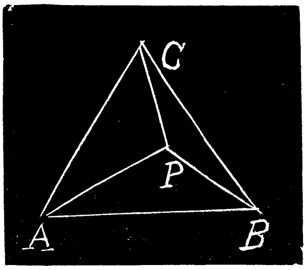
$$x+y=z, \quad y+z=x, \quad \text{or} \quad z+x=y,$$

i. e., one of the quantities x, y, z is equal to the sum of the other two.

This gives a solution of the problem. Therefore, construct a triangle of given species with its vertices on three concentric circles, radii x, y, z.

II. **Solution by J. SCHEFFRR, A. M., Hagerstown, Maryland.**

The following construction is valid for any triangle. Divide AB harmonically in the ratio $a:b$. On the two conjugate points construct a circle. Divide AC harmonically in the ratio $a:c$, and construct a circle on the line connecting the two conjugate points. The points common to the two circles satisfy the condition; for it is a well known theorem in geometry, that the circle erected on the distance of the two harmonic points of a straight line is the locus of the point whose distances from the extremities of the line are in the ratio in which the line is divided harmonically. There are consequently in general two points. The limitation is that the two circles either intersect or touch each other.



Treating the problem algebraically, let $AB=AC=BC=m$, $AP=ax$, $BP=bx$, $CP=cx$. Denoting $\angle PAB$ by θ, we have

$$\cos\theta=\frac{a^2x^2+m^2-b^2x^2}{2amx}, \quad \cos(60^\circ-\theta)=\frac{a^2x^2+m^2-c^2x^2}{2amx};$$

$$\therefore \tfrac{1}{2}\frac{a^2x^2+m^2-c^2x^2}{2amx}+\tfrac{1}{2}\sqrt{3}\sqrt{1-\frac{(a^2x^2+m^2-b^2x^2)^2}{4a^2m^2x^2}}=\frac{a^2x^2+m^2-c^2x^2}{2amx}.$$

Removing the radical and making all necessary reductions we finally get

$$(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2)x^4-(a^2+b^2+c^2)m^2x^2=-m^4.$$

Also solved by M. A. Harding, H. Prime, C. N. Schmall, and H. C. Feemster.

NOTES AND NEWS.

About 130 pages of the part of the French mathematical encyclopedia which was issued in June, 1912, are devoted to contemporary researches on the theory of functions. The three main subjects treated are the theory of concrete sets of points, the theories of integration and of finding derivatives, and the development into series. As regards the theory of sets of points, it is observed on page 115 that, in a very general way, it might be said that the German and English writers devote most attention to the abstract theory of sets of points, while the French writers lay most stress on the applications of this subject in the theory of functions. M.

Alfred Ackermann-Teubner has given twenty thousand marks—about five thousand dollars—to the University of Leipzig, to establish a *mathematical prize.* The first award is to be made in 1914, and every two years thereafter until the surplus accumulations amount to sixty thousand marks. After this time the prize is to be awarded annually. The subjects for which the prize after 1914 is to be awarded are, in order, as follows:

1. History, philosophy, and teaching;
2. Mathematics, especially arithmetic and algebra;
3. Mechanics;
4. Mathematical physics;
5. Mathematics, especially analysis;
6. Astronomy and theory of errors;
7. Mathematics, especially geometry;
8. Applied mathematics, especially geodesy and geophysics.

The range of the subject matter is to be about that given in the large German mathematical encyclopedia, which is now being published by B. G. Teubner of Leipzig, Germany. M.

In the "Summary Report" on the teaching of mathematics in Japan, which was recently published, there is given, page 197, a list of third year courses in mathematics in the Tokio Imperial University. This is of interest as it indicates how advanced their higher courses in mathematics really are. Four courses, bearing the following general headings,—General Theory of functions, Theory of differential equations, Theory of numbers and algebra, Higher geometry,—are outlined as follows: Riemann's surface and

its connectivity, analysis situs, elliptic integrals, theory of invariants and covariants, Hermite's transformation, Jacobi's principle of transformation, numerical evaluation of the elliptic integral, Abel's theorem and its applications, transformation of theta-functions, Fuchs' theory of linear differential equations, Gauss' differential equations, integration of partial differential equations, Weierstrass' method of the calculus of variation, introduction to integral equations, theory of groups of finite order, Galois' theory of equations, select chapters from higher arithmetic and algebra, one or two of the different kinds of geometries, such as differential geometry, non-euclidean geometry, descriptive geometry, etc. These advanced courses throw considerable light on their elementary work in mathematics. M.

The July number of the *Mathematical Gazette*, containing portraits of the prominent living Cambridge mathematicians, Darwin, Larmor, Hobson, and Love, commences with an article by the well known writer on mathematical history, W. W. R. Ball. The article is entitled "The Cambridge School of Mathematics," and it divides the history of this school into six periods, viz., the mediaeval, the renaissance, the Newtonian, the eighteenth century, the nineteenth century, and the present period. It may be remembered that W. W. R. Ball wrote a book of 264 pages on "Mathematics at Cambridge" in 1889, and hence it may be assumed that he is especially well prepared to write an article on this subject. As the closing paragraph seems to be of especial interest, we quote it: "In this article I have not unnaturally avoided mentioning the work of those who fortunately are with us today, and for similar reasons I do not propose to say anything about the progress of the school in the opening years of the twentieth century. The reconstruction in 1909 of the Tripos, and the destruction of many of the distinctive features of the former scheme *must profoundly modify the future history* of Mathematics at Cambridge, and perhaps the long continued efforts to bring students into closer touch with professors and lecturers may be at last crowned with success. The change in the Tripos regulations has been accompanied by a curious alteration in the popular subjects, and today but few of the young graduates who desired the change are interesting themselves in those branches of applied mathematics once so generally studied, but rather are turning their attention to subjects like the theories of *functions* and *groups*. It is too early to say whether this is only a passing movement." M.

The Summer meeting of the American Mathematical Society was held at the University of Pennsylvania on September 10-12. An extended program of papers occupied the sessions for two days, and an excursion to the historic points of interest added to the general enjoyment of the members present. S.

The number of American representatives who attended the International Congress at Cambridge, England, was most gratifying, being greater than that of any other country except Great Britain. The number was eighty, Germany and France coming next with seventy and fifty-two, respectively. The University of Illinois and the University of Chicago had five representatives each. S.

The International Commission on the Teaching of Mathematics made its report at the Fifth International Congress of Mathematicians at Cambridge, England, in August. Reports were received from eighteen countries, and 150 separate reports were submitted. About fifty more are now in process of preparation, and others are contemplated by various countries. The Central Committee, consisting of Professor Klein (Göttingen), Sir George Greenhill (London), and Professor H. Fehr (Geneva), with Professor David Eugene Smith (New York) added, was continued in office for another period of four years. The American reports have been completed and may be obtained gratis by application to the Bureau of Education, Washington, D. C. It is probable that one or more reports, summarizing the large features of the reports of all other countries, will be prepared by the American Commission during the next four years, and that certain other special lines of work will be undertaken. The Central Committee contemplates holding three international conferences on teaching, the first in France in 1914, the second in Germany in 1915, and the third, with the next Congress, in Stockholm in 1916. A more extended report will be given in the next issue. S.

The *Bulletin* of the American Mathematical Society announces the following promotions and appointments: Professor R. D. Carmichael of the University of Indiana, has been promoted to an associate professorship of mathematics; Dr. E. W. Sheldon has been promoted to a professorship of mathematics in the University of Alberta; Dr. E. T. Bell of Columbia University, has been appointed instructor in mathematics at the University of Washington; Mr. R. B. Stone has been appointed instructor in mathematics at Purdue University; Dr. W. M. Smith of Lafayette College, has been appointed assistant professor of mathematics in the University of Oregon; Dr. R. G. D. Richardson has been promoted to an associate professorship at Brown University; and Mr. E. P. R. Duval of Princeton University, has been appointed assistant professor of mathematics in the University of Kansas. F.

This issue was mailed five weeks late, party due to the fact that Professer Marshall failed to receive the proof for his article, which was sent to him by mail in September. After more than a month, the proof was returned unclaimed.

THEODORE LOUIS DELAND.

By ARTEMAS MARTIN, LL. D., Editor and Publisher, Mathematical Magazine, 918 N Street, Washington, D. C.

Theodore L. DeLand was born at Kirtland, Oneida County, New York, June 28, 1841; he died suddenly September 1, 1911, of heart failure, at his residence, 1415 Newton Street, Washington, D. C.

Mr. DeLand was educated at Fort Edward Institute.

He removed from New York to Aurora, Illinois, from which place he was appointed to a position in the Treasury Department, and came to Washington in 1871.

In 1894 he was transferred to the Civil Service Commission to attend to the work of the Treasury Department there, but returned to the Department in 1898.

In 1906 he was made Examiner of Clerks of the Treasury Department slated for promotion.

Mr. DeLand was one of the best known clerks of the Treasury Department and held many responsible positions during his forty years of faithful service.

Mr. DeLand was a mathematician of marked ability, and for years a mathematical expert in the Treasury Department.

He was a valued contributor to a number of the mathematical periodicals of his day, among the most important of which may be mentioned *The Analyst*, DeMoines, Iowa, edited and published by Dr. Joel E. Hendricks; the *Mathematical Visitor*, and the *Mathematical Magazine*, edited and published by the writer of this article; and THE AMERICAN MATHEMATICAL MONTHLY.

Mr. DeLand was very skillful in handling problems relating to bonds and other Government securities, and to finite differences.

He solved the following problem, which had been proposed by him in a previous number, on page 169 of No. 6, Vol. I (1881), of the *Mathematical Visitor:* "In 1861 a 6-per-cent 20-year coin bond of the United States, interesr payable semi-annually, sold on the market for $0.891 on the dollar; what, on this basis, would have been the market value of a 4-per-cent 28-year coin bond of the United States, interest payable quarterly?"

Mr. DeLand contributed the following papers to the *Mathematical Magazine:* "The United States Bond Problem," published in Vol. II, No. 9 (January, 1895), pages 161-163; "The United States Sinking Fund," published in Vol. II, No. 12 (September, 1904), pages 273-274.

He is survived by a widow, one son, and two daughters.

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THE FIFTH INTERNATIONAL CONGRESS OF MATHEMATICIANS.*

From a Report by J. W. A. YOUNG, The University of Chicago.

The Fifth International Congress of Mathematicians was held at Cambridge, England, August 21 to 27, 1912. Meeting in a University which for rare beauty and charm of picturesque mediaeval buildings and exquisite gardens, can find a rival only in its sister university of Oxford; living in the rooms occupied in times past by generation after generation of the world's greatest savants, dining in halls of storied interest from whose crowded walls look down the likenesses of great men of many an age whom the world still delights to honor; royally entertained with that cordial hospitality for which the Englishman is so justly famed, the mathematicians who gathered at Cambridge from the four corners of the world lived through a unique week of their lives, and carried away with them a souvenir, never to be forgotten, of a Congress brilliant alike in historic and lovely surroundings, in elaborate social functions, and in number and value of the mathematical lectures, reports and papers presented.

The 670 members present (567 full members and 103 members of their families) were registered from twenty-seven countries, as follows:

Great Britain, 250; United States, 82; Germany, 70; France, 52; Italy, 38; Russia, 38; Spain, 25; Austria, 19; Hungary, 19; Sweden, 13; Holland, 9; Switzerland, 9; Denmark, 5; Greece, 5; Roumania, 5; Belgium, 4; Brazil, 4; Canada, 4; Norway, 4; India, 3; Japan, 3; Portugal, 3; Egypt, 2; Bulgaria, 1; Chili, 1; Mexico, 1; Servia, 1.

The sessions of the Congress were of three types: general sessions, lectures, and sectional meetings.

At the first general session Sir G. Greenhill made the following statein regard to the work of the International Commission on the Teaching of Mathematics:

* This article includes only those portions of Professor Young's report which refer particularly to the pedagogical aspects of the work of the Congress. A more general report will be found in the *Bulletin of the American Mathematical Society*. EDITORS.

"The International Commission on the Teaching of Mathematics was appointed at the Rome Congress, on the recommendation of the members of Section IV. The several countries, in one way or another, have recognized officially the work, and have contributed financial support. About 150 reports have been published, and about 50 more will appear later. The Commission will report in certain Sessions of Section IV. The Commission hopes to be continued in power, in order that the work now in progress may be brought to completion."

At the third general session, which closed the Congress, the following resolution was unanimously passed:

Resolved, That the Congress expresses its appreciation of the support given to its Commission on the Teaching of Mathematics by various governments, institutions, and individuals; that the Central Committee composed of F. Klein (Göttingen), Sir G. Greenhill (London), and H. Fehr (Geneva), be continued in power and that, at its request, David Eugene Smith of New York be added to its number; that the delegates be requested to continue their good offices in securing the coöperation of their respective governments, and in carrying on the work; and that the Commission be requested to make such further report at the Sixth International Congress, and to hold such conferences in the meantime, as the circumstances warrant.

The sectional work was carried on in the following sections:

I. Arithmetic, Algebra, Analysis.
II. Geometry.
IIIa. Mechanics, Physical Mathematics, Astronomy.
IIIb. Economics, Actuarial Science, Statistics.
IVa. Philosophy and History.
IVb. Didactics.

The International Commission on the Teaching of Mathematics held one session separately, and three sessions jointly with Section IVb.

There were twenty-eight papers listed in Section I, twenty-two papers in Section II, eighteen papers in Section IIIa, eleven papers in Section IIIb, and twenty-one papers in Section IVa.

Section IVb held six meetings, two of them jointly with Section IVa. Three of these sessions were held jointly with the International Commission for the Teaching of Mathematics. At the first of these sessions the following address on the work of the Commission was delivered by David Eugene Smith, who had been in recent conference with the President of the Commission, Professor Klein, and the Central Committee:

"As has already been mentioned, Professor Klein, to whose great energy and wisdom the success of the International Commission on the Teaching of Mathematics is largely due, is unable to be present, on account of illness. It was my privilege to propose to the delegates at our meeting on Wednesday the sending of a telegram to Professor Klein, and I now propose the same message to Section IV, as follows: 'The International Com-

mission on the Teaching of Mathematics, and Section IV, at their first Cambridge Meeting, express regret at your absence and best wishes for your recovery.'

"The Commission was organized for the purpose of reporting upon the present status of the teaching of mathematics in the various countries of the world. Special sub-committees have also been appointed from time to time, to consider questions of international rather than merely national interest. About one hundred and fifty reports on the work done in the various countries have been prepared, and at least fifty more are in contemplation. A world-wide interest in the improvement of mathematical teaching has been awakened, and the influence of the movement is certain to be very far reaching. Ten countries have completed the task set for themselves. In chronological order of completion these countries are Sweden, Holland, France, Switzerland, Austria, Japan, the United States of America, the British Isles, Hungary, and Denmark. In process of publication is the monumental work of Germany, with twenty-seven out of thirty-six reports already printed, and the reports of Italy, Roumania, Spain, and Russia. In contemplation are the reports of Greece, Norway, Australia, Portugal, Servia, and doubtless of several other countries.

"As to the future work of the Commission, the Central Committee earnestly desires that it be authorized to see to the completion of the reports. It is therefore very desirable that it be continued in power, both for this purpose and for the consideration of certain questions of great international significance. Such topics as the proper training of engineers, of calculus in the secondary schools, of the general value of intuition in the teaching of mathematics, of the training of teachers, and of the educational (cultural, disciplinary, non-technical) value of mathematics, may properly occupy the attention of the Commission in the next four years. Special conferences having already been held, at Bruxelles and Milano, it is proposed, if the Committee is continued in power, to hold others between now and the time of the meeting of the Congress in 1916, if that shall be the date. Possibly such conferences may be held in France in 1914, in Germany in 1915, and in Stockholm in 1916.

"It is also hoped that each country will prepare a summary of the large features of the reports of other countries, to the end that the work that has been accomplished may have its full effect. It is further hoped that the various countries will continue the financial support that has been given to the Central Committee in the past.

"A word should be said at this time in memory of those distinguished teachers who have been connected with the movement, but who have been called from their labors to solve the Great Problem. Soon after the last Congress adjourned, Professor Vailati of Rome, a distinguished writer and an accomplished scholar, passed away. Scarcely in his full prime of life, his loss is felt not by Italy alone, but by all who appreciate scholarship

and high educational standards Professor Bovey, president of the Imperial Technical College at South Kensington, and who was charged with the labor of reporting for Canada, has also been called from us. In his death the world lost a scholar and an administrator of prominence. And as he was planning to attend this Congress, four weeks ago today, Geheimrath Professor P. Treutlein of Carlsruhe, passed suddenly away. In his death Germany lost one of her foremost educators, and the International Commission, one of its best supporters.

"The Central Committee has consulted with the Committee on Organization and it has been decided that the first set of reports shall be presented to the Library of the University of Cambridge, a second set to our official hosts, the Cambridge Philosophical Society, and a third set to that great world-library, the Library of the British Museum."

The General Secretary of the Commission next made a statement as to the work of the Central Committee, and submitted its collected publications.

Thereupon the reports of the various countries were formally submitted to the Congress. The countries were called in alphabetical order in the French language, and the following members of the Commission presented the reports, with a brief oral description, and a longer written statement, which will be published in the Proceedings of the Congress, and in the official organ of the Commission, *L'enseignement mathématique.* The statement accompanying the American report will be published in *School Science and Mathematics*, the official American organ.

Germany, Prof. A. Gutzmer (Halle); Austra, Prof. E. Czuber (Vienna); Belgium, Principal E. Clevers (Ghent); Denmark, Prof. H. Fehr; Spain, Prof. Toledo (Madrid); United States, Prof. J. W. A. Young (Chicago); France, Prof. C. Bourlet (Paris); Greece, Prof. H. Fehr; Holland, Prof. J. Cardinaal (Delft); Hungary, Prof. E. Beke (Buda-Pesth); British Isles, Prof. C. S. Jackson (Woolwich); Italy, Prof. G. Castelnuovo (Rome); Japan, Prof. R. Fujisawa (Tokio); Norway, Prof. M. Alfsen (Christiania); Portugal, Prof. F. J. Teixeira (Oporto); Roumania, Prof. G. Tzitzeica (Bucharest); Russia, Prof. H. Fehr; Sweden, Prof. H. Fehr; Switzerland, Prof. H. Fehr (Geneva).

Also the following associated countries: Brazil, Prof. E. de B. R. Gabaglia (Rio de Janeiro); Servia, Prof. M. Petrovitch (Belgrade).

At the second joint session of Section IVb and the International Commission, the report of sub-commission B, on "The mathematical education of the physicist in the university," was presented by Professor C. Runge, and followed by a lively discussion.

At the last joint session of Section IVb and the International Commission, C. Goldziher presented a report on the work done by David Eugene Smith and himself towards preparing a bibliography of works on the teaching of mathematics, published since 1900. (This bibliography is about to be

published by the United States Bureau of Education and can be obtained from the Bureau on request.) Upon motion of Professor Smith the following resolution was passed:

"*Resolved*, That Section IVb of the International Congress of Mathematicians, assembled at Cambridge, expresses its thanks to the Honorable the United States Commissioner of Education for his great interest in publishing, for free distribution, the recent bibliography on the teaching of mathematics (1900-1912), and the hope that it may, through his good offices, be brought to completion to the year 1915, with such additions to the present list as may seem desirable."

David Eugene Smith then presented the Report of Sub-commission A on "Intuition and experiment in mathematical teaching in secondary schools." The presentation of the report was followed by an extended discussion. This report will be published in the various official organs named above.

In the other sessions of Section IVb, the following papers were presented:

Whitehead, A. N. "The principles of mathematics in relation to elementary teaching."

Suppandschitch, R. "Le raisonnement logique dans l'enseignement mathématique universitaire et secondaire."

Hill, M. J. M. "The teaching of the theory of proportion."

Hatzidakis, N. "Systematische Recreationsmathematik in den mittleren Schulen."

Gérardin, A. "Sur quelques nouvelles machines algébriques."

Carson, G. St. L. "The place of deduction in elementary mechanics."

Nunn, T. P. "The proper scope and method of instruction in the calculus in schools."

It was not possible to secure brief abstracts of the above papers for incorporation in this report. The papers will be published in full in the Proceedings of the Congress, and elsewhere.

This account of the Congress would be lamentably incomplete without brief mention of its social side. First of all, the four official receptions, that by Sir G. H. Darwin, President of the Cambridge Philosophical Society, in St. John's College, on Wednesday evening; that by Lord Rayleigh, Chancellor of the University, in the Fitzwilliam Museum, on Friday evening; that by the President of the Congress in Christ's College, on Sunday afternoon; and finally that by the Master and Fellows of Trinity College on Monday evening. These brilliant functions, each in a unique setting with a charm all its own, will remain in memory as pictures not to be forgotten. Sunday was fittingly closed with an organ recital in King's College Chapel, the most lustrous of all of Cambridge's architectural gems. Visits to the Observatory, and to the Cambridge Scientific works, with attendant teas, excursions to Ely, to Oxford, and to Hatfield House, were temptations that caused the de-

votion to his science of more than one mathematician temporarily to waver. Besides all this an able and enterprising committee of ladies had prepared for the ladies of the Congress a most attractive series of visits to the Colleges, of drives and teas in Cambridge and its environs, and of excursions to various points of interest that seemed to leave no moment without something tempting to do. Whether viewed from the social side or from the mathematical side, the Congress must be pronounced a complete success.

THE WORD "RADIAN."

By A. R. CRATHORNE, University of Illinois.

Few terms in elementary mathematics are of such recent origin that the time and place of their first introduction into the literature can be determined. The word "radian" seems to be an exception to this statement, however, for its history goes back some forty years only. One of the originators of the word is still living.

When the "R" volume of Murray's "New English Dictionary" appeared, it gave under the definition of "radian" the date 1879, and a reference to the *Treatise on Natural Philosophy,* by Thomson and Tait, from which one inferred that here was the first use of the word. The derivation was given as "radius+an." This statement in the dictionary called forth a letter to *Nature* (Vol. 83, p. 156) from Professor Thomas Muir of Cape Town, South Africa, who claimed to have used the term in his classes at St. Andrew's University as early as 1869. At that time he was hesitating between the three words "rad," "radial" and "radian" with leanings towards the monysyllable. In 1874 after discussing the matter with the late Professor James Thompson and with Alexander Ellis, he finally adopted the term "radian," considering it as a contraction of "radial angle."

In answer to the letter of Professor Muir, Mr. James Thomson, son of Professor James Thomson, wrote a letter to *Nature* (Vol. 83, p. 217), in which he pointed out that the word had been used in 1871 by his father, before the latter knew Dr. Muir, and that on June 5, 1873, the word was printed in the examination questions at Queen's College, Belfast. These questions were published in the college calendar.

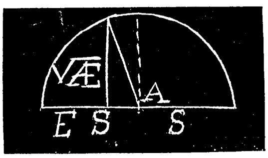
Two more letters from Mr. Thomson and Dr. Muir on the subject appear in the same volume of *Nature*, pp. 459, 460, from which it is seen that the term "radian" was used independently for some years by Professor Muir and Professor James Thomson, but that the first printed word was in the above mentioned examination papers. Professor Muir also originated the physical term "therm" in 1870 (see *Nature*, Vol. I, p. 606).

HISTORICAL NOTE ON THE GRAPHIC REPRESENTATION OF IMAGINARIES BEFORE THE TIME OF WESSEL.

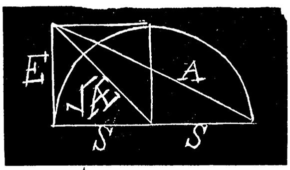
By FLORIAN CAJORI, Colorado College.

John Wallis's attempts at graphic representation of imaginaries have been described by H. Hankel,* W. W. Beman,† D. E. Smith,‡ and more fully by G. Eneström.§ They refer to Wallis's *Treatise of Algebra*, London, 1685, pp. 264-273, but none of them mention Wallis's earlier discussion of this subject in a letter to Collins, May 6, 1673, where he suggests a construction a little different from any of the constructions found in his *Algebra*. This letter was written three or four years before the manuscript of his *Algebra* was ready for print. Wallis says in the letter:‖

"This imaginable root in a quadratic equation I have had thoughts long since of designing geometrically, and have had several projects to that purpose. One of them was this: Supposing a quadratic equation $2SA-A^2=\text{Æ}$, or (which is equivalent) $A^2-2SA+\text{Æ}=0$. If $S\ (=\frac{A+E}{2})$ be bigger than $\sqrt{(\text{Æ})}$; that is $S^2>\text{Æ}$, the roots are $S\pm\sqrt{(S^2-\text{Æ})}=\left\{\begin{matrix}A\\E\end{matrix}\right.$, putting . . . $S=\frac{1}{2}Z=\frac{A+E}{2}$ and . . . $V=\frac{1}{2}X=\frac{A-E}{2}$, where $V\ [=\sqrt{(S^2-\text{Æ})}]$, added to and taken from S, yields $S+V=A$, $S-V=E$, that is, [the roots are] $S\pm\sqrt{(+V^2)}$.



"But if Æ be bigger than S^2, the roots are $S\pm\sqrt{(S^2-\text{Æ})}\ (=S\pm\sqrt{(-V^2)})$, where $\sqrt{(\text{Æ})}$, which was the sine, now becomes the secant, and V, that was the cosine, is now the tangent. For $S^2\sim\text{Æ}=V^2$, the difference of the planes S^2 and Æ, the greater is to be expressed by the hypotenuse, and the less by the perpendicular."



Evidently, S and Æ are here always positive, hence this is not a general construction of the roots of the quadratic equation.

In both figures the lines E and A represent the roots of the quadratic. In both, the line E extends from the left end of the diameter to the terminal of the line V; the line A begins where E ended and extends to the right end of the diameter. Thus the analogy in the construction of real roots in the first figure and of complex roots in the second figure is complete. More-

* Hankel. *Complexe Zahlen*, Leipzig, 1867, pp. 81, 82.

† *Proceedings of the American Association for the Advancement of Science*, Vol. 46, 1897, pp. 35, 36.

‡ Merriman and Woodward, *Higher Mathematics*, 1898, pp. 515, 516.

§ *Bibliotheca Mathematica*, 3rd S., Vol. 7, pp. 263-269.

‖ S. J. Rigaud, *Correspondence of Scientific Men of the Seventeenth Century*, Vol. 2, Oxford, 1841, p. 578.

over there is an attempt to secure vector addition. We have $E+A=2S$, also $a+ib+a+ib=2a+2ib$; but we do not have $a+ib+c+id=(a+c)+i(b+d)$. This method of geometric representation of imaginaries labors under another fatal defect, that of representing conjugate roots by lines of different lengths. While this method is less interesting than some others found later in Wallis's *Algebra*, it has one point of superiority over them in being perfectly determinate. Eneström has shown that some of Wallis's geometric notions permit the vector standing for a complex root to take any one of an indefinite number of different directions.* It will be noticed that the present method fails in case of the pure imaginary $\sqrt{(-1)}$, since for that case the radius S of the circle vanishes.

It is well known now that Kühn, in 1750, did not attempt a geometric picture of an imaginary, but simply tried to interpret a negative plane.† He was anticipated by Wallis, who touches upon this point not only in his *Algebra*, but also in the letter of 1673, from which we have been quoting. Wallis says in the letter:

"I was of opinion from the first, that a negative plane may as well be admitted in algebra as a negative length, both being in nature equally impossible; for there can no more be a line less than nothing than a plane less than nothing, both being but imaginable; and if we suppose such a negative square, we may as well suppose it to have a side, not indeed an affirmative, or a negative length, but a supposed mean proportional between a negative and positive thus designable, $\sqrt{(-n)}$, or rather $\sqrt{(-n^2)}$, that is, $\sqrt{(+nx^x-n)}$, a mean proportional between $+n$ and $-n$."

No doubt Wallis touched upon geometric notions on imaginaries in other letters. There is a reference to this subject in one of September 11, 1676.‡ Before this, Collins expressed himself in a letter to J. Gregory, October 19, 1675, as follows:§

"I nothing doubt but the roots of such negative squares, etc., denote an impossibility; as for instance Dr. Wallis, in his first tome, assumes the two legs of a triangle, 2 and 1, to be less than the base 4, and that to show that algebra might be fallacious to a tire, and really finds the segments of the base well; but had he proceeded further he would have found the perpendicular to have been $\sqrt{(-105)}$, which had minifested the impossibility."

A few years ago Felix Müller made the statement that Euler had given a geometric representation of the imaginary by means of a circle.|| Eneström has explained the construction¶ and pointed out that it is the same as one given a hundred years earlier by Wallis, and is in fact trivial. It is quite possible that Euler, like Wallis, may have entertained different

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* *Bibliotheca Mathematica*, 3rd S, Vol. 7, p. 266.

† Eneström in *Bibliotheca Mathematica*, 3rd S, Vol. 7, pp. 263-269.

‡ Rigaud, *op. cit.*, Vol. 2, p. 594.

§ Rigaud, *op. cit.*, Vol. 2, p. 278.

|| *Festschr. z. Feier d. 200. Geburtstages L. Eulers*, Leipzig und Berlin, 1907, p. 96; *Opusc. analyt.*, Vol. 2, pp. 76-90.

¶ *Bibliotheca Mathematica*, 3rd S., Vol. 9, 1908-1909, p. 182.

schemes. The query has arisen in the mind of the writer whether not only Euler but also Charles Walmesley had the Gaussian representation in mind when writing the passages we are about to quote. In Euler's great article, *De la controverse entre Leibnitz et Bernoulli*, etc., on the logarithms of complex numbers,* it is explained under "Probleme 4" how to find the anti-logarithm, when the logarithm $g\sqrt{(-1)}$ is given. Euler says:

"... pour le trouver on n'a qu'a prendre un arc de cercle$=g$, le rayon etant$=1$ et ayant cherche son sinus et cosinus, le nombre cherche sera $x=\cos g+\sqrt{(-1)}\sin g$."

Much the same language is used by Euler in another article in the same volume of the Berlin memoirs (p. 278), *Sur les racines imaginaires des equations.* Six years later Charles Walmesley, a Roman Catholic prelate, wrote an article on logarithms.† To find the logarithm of $a+b\sqrt{(-1)}$, he changes it to the form $(a^2+b^2)^{-\frac{1}{2}}(y+u\sqrt{(-1)})$, and says, "il est clair qu'on doit prendre l'arc de cercle dont le sinus est u, le cosinus y," etc.

Did Walmesley and Euler, in connection with $y+ui$, carry in their minds the geometric picture of the lines that weie generally used at that time and long before to represent u the sine and y the cosine, the two lines being perpendicular to each other? In other words, did Euler and Walmesley have in mind the Wessel-Argand diagram? There is danger of reading into passages ideas which the authors had not entertained. Hence we must wait for more careful studies of other articles by the same authors, before a final conclusion can be reached.

Another conjecture presents itself. It is asserted that Gauss arrived at the Wessel-Argand diagram independently of Wessel and Argand. We know that Gauss in his doctor's dissertation of 1799 passes in critical review the eighteenth century proofs of the theorem that every equation has a root, including Euler's proof‡ contained in the article *Sur les racines imaginaires des equations*, named above. Our conjecture is that the diagram suggested itself to Gauss upon the reading of the passages in Euler, quoted above.

That the Wessel-Argand diagram was probably in the minds of certain eighteenth century mathematicians before the time of Wessel appears also from a remark of Cauchy,§ to the effect that "a modest scholar," Henri Dominique Truel, as early as the year 1786 represented imaginary quantities upon a line perpendicular to the line for real quantities. We are not aware that Truel ever published his results.

The most interesting and valuable graphic representation of imaginaries before Wessel has never been described by historians of mathematics. It is accomplished by means of a circle and an equilateral hyperbola. Wallis in his *Algebra* suggests various constructions of imaginaries. One of them represents $\sqrt{(1-x^2)}$ as the ordinate of the circle $x^2+y^2=1$ when $x<1$, and

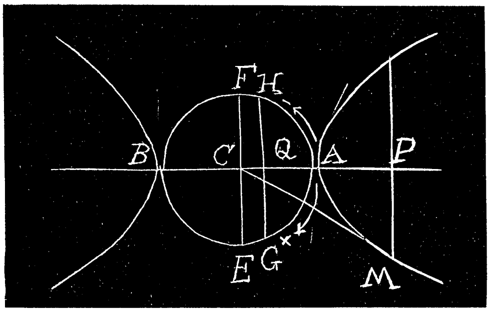
* *Histoire de l'academie r. d. sciences et b. l.*, année 1749, Berlin, 1851, pp. 139-179.

† *Histoire d. l'academie r. d. sciences et b. l.*, année 1755, Berlin, 1757, p. 397.

‡ Ostwald's *Klassiker der Exakt. Wiss.*, No. 14, Leipzig, 1890, pp. 13-21.

§ Cauchy, *Exercises d'analyse et de phys. math.*, T. iv, 1847, p. 157.

as the ordinate of the hyperbola $x^2-y^2=1$ when $x>1$. During the eighteenth century one encounters not infrequently statements to the effect that real (imaginary) arcs of the circle are imaginary (real) arcs of the hyperbola.* These ideas led W. J. G. Karsten in 1768† to the invention of a diagram displaying the infinitely many logarithms of a real or a complex number. He remarks that all ordinates of the circle $x^2+z^2=1$ are imaginary ordinates of the hyperbola $x^2-y^2=1$, where $y=\sqrt{(-1)}z$, that therefore the circle may be regarded as an imaginary part of the hyperbola, and *vice versa.* Consider each of the four arcs coming together at A, or at B, as the continuation of each of the other three arcs. An arc AG for the circle may be considered as an imaginary arc of the hyperbola. Between any two points M and G exist numberless different arcs, MAG, $MAGEBFG$, or in general, $MAG+2\lambda\pi$, and also $MAHBG+2\lambda\pi$, where $\lambda=0, 1, 2, \ldots$ This is true even if M, or G, or both, coincide with A. To any abscissa x there belong therefore not only numberless arcs, but also numberless corresponding sectors. By the Calculus, double the area of a hyperbolic sector corresponding to the abscissa x is $=\log(x+y)=\sqrt{(-1)}.\text{arc.cos}x$, the sector being assumed zero when $x=1$. If $x>1$, say $x=CP$, then $\log(x+y)$ gives double the sectorial areas corresponding to the arcs $AM\pm\lambda\ (AGBFA)$, that is, $AM\pm2\lambda\pi\sqrt{(-1)}$. Only the sector ACM is real. If $x<1$, then $x+y$ is imaginary. Let $x=CQ$, then $\log(x+y)$ equals double the sectorial areas whose arcs are $AG\pm2\lambda\pi\sqrt{(-1)}$, or $AFBG\pm2\lambda\pi\sqrt{(-1)}$. All of these arcs and sectors are imaginary. From this we see that $\log(-1)$ is represented by double the areas of the sectors belonging to arcs $AEB+\lambda\ (BFAEB)$, or also $AFB+\lambda\ (BEAFB)$.



This graph is in accordance with Euler's formula $\log(-1)=(2\lambda+1)\pi\sqrt{(-1)}$. When $x=0$ and $\lambda=0$, we get the particular logarithm of $\log\sqrt{(-1)}$ which is represented by twice ACE. Here then Karsten gives a graphic representation of the well-known Bernoulli-Euler expression $\log\sqrt{(-1)}=\frac{\pi}{2}\sqrt{(-1)}$. In 1842 DeMorgan remarked that "not many years since, [it] was one of the mysteries of analysis," and he apparently believed that he himself had given the earliest geometric interpretation of it.‡ Karsten's diagram gives a geometric picture of the natural logarithms of any real or imaginary number. He also considers the possibility of a graphic representation by using, in place of sec-

* See for instance, Vincenzo Riccati *Sopra logaritmi dei numeri negativi lettere cinque,* Modena, 1779, p. 66. John Playfair in *Phil. Trans.*, Vol. 68, year 1778, Pt. i, pp. 318-343.

† "Abhandlung von den Logarithmen verneinter Grössen," 1. und 2. Abtheil., *Abh. Münch. Akad.* V, 1768

‡ *Transactions of the Cambridge Philosophical Society,* Vol. 7, 1842, p. 294.

tors, the hyperbolic trapezoids (the parallel sides being parallel to one of the asymptotes, the other two sides being the hyperbolic arc and the other asymptote), but he finds this less general and less convenient. Nowhere, either in eighteenth century or nineteenth century authors have we been able to find a reference to Karsten's geometric construction of imaginary logarithms.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

374. Proposed by H. PRIME, Boston, Massachusetts.

Divide an angle of 30° into two parts so that the product of the third and fourth powers of their sines (or cosines) shall be a maximum. To be solved without using the methods of calculus. [From *The Maine Farmers' Almanac*, 1912.]

Solution by H. E. TREFETHEN, Colby College.

Let x and $30°-x$ be the two parts. $\sin^4 x \sin^3(30°-x)=\text{maximum}$, when $\sin^{\frac{4}{3}} x \sin(30°-x)=\text{maximum}$, or when $\sin^{\frac{4}{3}} x \cos x-\sin^{\frac{7}{3}} x \sqrt{3}=\text{maximum}$. Put $\sin^{\frac{1}{3}} x=y$, $\cos x=(1-y^6)^{\frac{1}{2}}$; and then $y^4(1-y^6)^{\frac{1}{2}} - y^7\sqrt{3}=m$, whence

$$y^{14}-y^8/4+my^7\sqrt{3}/2+m^2/4=0 \ldots (1).$$

Let c be a value of y that renders m a maximum. Then the first member of (1) must be exactly divisible *twice* by $y-c$, since a maximum or minimum corresponds to two equal roots. The quotient is readily written down by the synthetic process. The first remainder$=c^{14}-c^8/4+mc^7\sqrt{3}/2+m^2/4=0$, since the division must be exact. Also the second remainder$=14c^{13}-2c^7+7mc^6\sqrt{3}/2=0$.

Eliminating m from the last two equations we get $196c^{14}-203c^8+16c^2=0$. Whence $c^2=0$ corresponding to a minimum, and $196c^{12}-203c^6+16=0$, $c^6=y^6=\sin^2 x=4.813227/56$, or $53.186773/56$. The former gives $x=17° \ 2' \ 52.9''$; the latter ($=\cos^2 x$) gives $x=12° \ 57' \ 7.1''$, and $30°-x=12° \ 57' \ 7.1''$ or $17° \ 2' \ 52.9''$.

It is to be noted that this method for determining maxima and minima can be applied to *any algebraic* expression to which the methods of calculus are applicable.

Also solved by J. Scheffer and A. H. Holmes.

375. Proposed by S. LEFSCHETZ, University of Nebraska.

Prove that $\frac{e}{2m+2} < e - (1+\frac{1}{m})^m < \frac{e}{2m+1}$. [Schlömilch.]

Solution by H. E. TREFETHEN, Colby College.

Dividing by e and subtracting each member from unity, we deduce

$$1/(1+1/2m) < (1+1/m)^m/e < (1+1/2m)/(1+1/m) \ldots (1).$$

(i) Let n be positive, and put $m=-n$. Then (1) becomes

$$1/(1-1/2n) < (1-1/n)^{-n}/e < (1-1/2n)/(1-1/n) \ldots (2).$$

$$-\log(1-1/2n) < -n\log(1-1/n) - 1 < \log(1-1/2n) - \log(1-1/n) \text{ or}$$
$$1/2n + 1/2^2.2n^2 + 1/2^3.3n^3 + \ldots + 1/2^r rn^r$$
$$< 1/2n + 1/3n^2 + 1/4n^3 + \ldots + 1/(r+1)n^r$$
$$< 1/2n + (2^2-1)/2^2.2n^2 + (2^3-1)/2^3.3n^3 + \ldots + (2^r-1)/2^r rn^r.$$

These series all converge when $n>1$. We also have

$1/2^r rn^r < 1/(r+1)n^r < (2^r-1)/2^r rn^r$.
$2^r > (r+1)/r > 2^r/(2^r-1$, $\quad 2^r-1 > 1/r > 1/(2^r-1)$.
For $2^r-1 > r > 1/r$ when $r \geqq 2$ (r being a positive integer).

Thus the given relations are established for the case when $n>1$, $-n < -1$, that is when $m<-1$.

(ii) In (1) put $m=n-1$. Then after multiplying by $n/(n-1)$ we find

$$1/(1-1/2n) < (1-1/n)^{-n}/e < (1-1/2n)/(1-1/n).$$

This is the same as (2) in (i) and the case is proved when $n>1$, $n-1 > 0$, that is when $m>0$.

(iii) If $m=0$, the given expressions become $e/2 < e-1 < e$ and the case is proved for $m=0$.

(iv) But in the interval $0 > m \geqq -1$ there are various results for different values of m.

(a) For such values of m as render the given functions real and finite, the given inequalities are true if $0>m>1/2$, but must be reversed if $-1/2 > m > -1$.

(b) If $m=-1/2, -1/4, -3/4, \ldots, -\frac{[2p-1]_{p=1}^{p=q}}{2q}$, $(1+\frac{1}{m})^m$ is imaginary.

(c) If $m=-1/2$, $e<e+\sqrt{(-1)}<\infty$.

(d) If $m=-1$, $\infty<e-\infty<-e$.

Therefore the given statements are proved true unless $0>m\overline{\gtrless}-1$. For case (iv) no general statement of relative magnitudes can be made on account of discontinuous functions.

376. Proposed by W. W. BEMAN, Professor of Mathematics, University of Michigan, Ann Arbor, Michigan.

If $\frac{(1+1/m)^m}{e}=1-a_1\frac{1}{m}+a_2\frac{1}{m^2}-a_3\frac{1}{m^3}+\ldots$, prove $na_n=\sum_{k=1}^{k=n}\frac{k}{k+1}a_{n-k}$, and compute $a_1, a_2, a_3, \ldots, a_8$.

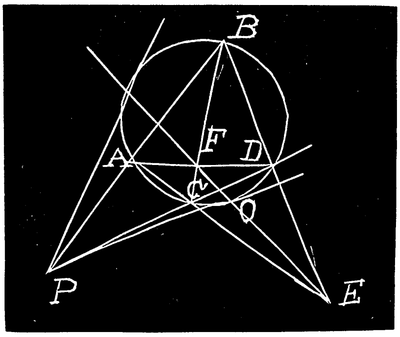
No solution of this problem has been received.

GEOMETRY.

400. Proposed by FRANCIS RUST, C. E., Pittsburgh, Pennsylvania.

Given a circle and a point P without; construct, *using the straight edge only*, the two tangents to the circle through P.

II. Solution by GEORGE W. HARTWELL, Hamline University, St. Paul, Minnesota.



Through P draw any two secants AB and CD, cutting the circle in A, B, C, and D. Join A and D, and A and C; B and D, and B and C. AC and BD meet at E, and AD and BC meet at F. Join E and F. EF is the polar of P. Then the points O and M in which the line EF intersects the circle are the points of tangency.

402. Proposed by H. PRIME, Boston, Mass.

The diameter of a hoop-shaped ring (or collar) is 24 inches at one edge and 28 inches at the other edge. A cross-section is a crescent with circular arcs of 120° and 60°, whose common chord is 4 inches long. Find its volume by elementary methods (without the use of calculus or the center of gravity).

Solution by H. E. TREFETHEN, Colby College.

Denote the given chord by AB, the axis of the ring by QQ, the arc of 120° by s, of 60′ by s'. Let ABC be an equilateral triangle. Complete the arcs s and s', and through A and C draw their diameters parallel to QQ.

Then we have $QA=12$, $QC=10$, 4=radius of arc s', $4\sqrt{3}/3$=radius of arc s. $8\pi/3-4\sqrt{3}$=area of segment ABs', $16\pi/9-4\sqrt{3}/3$=area of ABs. $v=16.2\sqrt{3}.\pi/6$=volume generated by either segment revolving about its diameter, $2\sqrt{3}$ being the projection of AB on the axis.

Put V and V' for the volumes generated respectively by the segments ABs and ABs' revolving about QQ. Then apply the theorem: If a plane figure exterior to two parallel lines in its plane revolve in succession about each of them as axes, the difference between the volumes of the solids thus generated is equal to the area of the generating figure multiplied by 2π times the distance between the axes. Thus $V-v=(16\pi/9-4\sqrt{3}/3).2\pi.12$, $V'-v=(8\pi/3-4\sqrt{3}).2\pi.10$, $V-V'=16\pi(9\sqrt{3}-2\pi)/3=155.9113$ cubic inches= required volume.

PROBLEMS FOR SOLUTION.

ALGEBRA.

380. **Proposed by J. K. ELWOOD, Superintendent Lucas Public Schools, Lucas, Kansas.**

A and B set out to walk around a cinder path a mile in circumference, and walk 3 hours, A walking 8 miles farther than B. Each reduced his rate one mile per hour at the end of the first hour, and again one mile per hour at the end of the second hour, his speed being otherwise uniform. They start in the same direction, but 12 minutes after A has passed B the third time he turns and walks in the other direction until 6 minutes after he has met B the third time, when he returns to his original direction, and overtakes B four times more. Determine their initial velocities. Illustrate by a time-table.*

381. **Proposed by S. A. COREY, Hiteman, Iowa.**

A, B, and C simultaneously make assignments of their property for the benefit of their creditors. The assets of A, B, and C were d, e, and q, respectively.

A's indebtedness as principal was a; A's indebtedness as surety for B was g; A's indebtedness as surety for C was h; A's indebtedness as surety for B and C jointly was i, (*i. e.*, B was surety for C, and A, in turn, was surety for B).

B's indebtedness as principal was j; B's indebtedness as surety for A was k; B's indebtedness as surety for C was l.

C's indebtedness as principal was m; C's indebtedness as surety for A was n; C's indebtedness as surety for B was p.

The law requires the surety to pay only such a portion of the debt as his principal cannot pay. What is the amount of the legal indebtness of each?

382. **Proposed by C. E. FLANAGAN, Wheeling, West Virginia.**

A few days ago I deduced the following formula for finding the value of the unknown quantity in a cubic equation having the form: $x^3+3A^2x=B$.

Let $C=\sqrt{\dfrac{2B}{A^3}+1}-1$. Then, $x=\dfrac{2B}{A^2(C^2+12)}+\dfrac{AC}{4}$.

* This problem is a variation of one by Todhunter. George H. Taber, of Pittsburgh, Pennsylvania, is responsible for it.

It is required to show: (1) How this formula was derived; (2) Under what conditions does it give exact results; (2) In general, what is its degree of approximation; (4) If possible, modify it so that it will always give exact results; (5) If it cannot be so modified show why.

GEOMETRY.

409. Proposed by S. LEFSCHETZ, Ph. D., University of Nebraska.

The sides of a triangle being in arithmetic progression, a and a' being, respectively, the smallest and largest sides; r, R the radii of the inscribed and circumscribed circles. To prove that $6Rr=aa'$.

410. Proposed by A. H. HOLMES, Brunswick, Maine.

Given a focus and two tangents to an ellipse, prove that the locus of the foot of the normal corresponding to either tangent is a straight line.

411. Proposed by C. N. SCHMALL, New York City.

ABCD is a rectangle of known sides. *BC* being produced indefinitely, it is required to draw a straight line from *A* cutting *CD* and *BC* in *X* and *Y*, respectively, so that the intercept *XY* may be equal to a given straight line.* (Unsolved in *Educational Times.*)

CALCULUS.

331. Proposed by T. H. GRONWALL, Ph. D., C. E., 912 Schiller Building, Chicago, Illinois.

To show that:

$$(1)\ \frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)=\frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y+\frac{n+1}{2}\pi\right)dy,$$

$$(2)\ \frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right)=\frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y+\frac{n}{2}\pi\right)dy.$$

332. Proposed by WILLIAM MARSHALL, Purdue University.

Find the exact or approximate value of the integrals, (a is constant),

$$(1)\ \int_1^\infty \sin(2x+a\log x)\frac{dx}{x^{3/4}};\qquad (2)\ \int_1^\infty \cos(2x+a\log x)\frac{dx}{x^{3/4}}.$$

333. Proposed by C. N. SCHMALL, New York City.

Evaluate $\iiint\sqrt{\dfrac{1-(x^2+y^2+z^2)}{1+x^2+y^2+z^2}}\,dx\,dy\,dz$, where $x^2+y^2+z^2>1$.

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* This problem is No. 16,766 in the *Educational Times*. It was proposed by D. Biddle, who, finding that no solutions of it appeared after a time, gave himself what he calls "An Easy Method of Approximating to a Correct Solution." This appeared in the issue of April 1, 1912, page 180. Mr. Biddle gives a few remarks on the problem in which he says that it is one of the many problems discussed in a paper by Prof. R. C. Archibald, of Brown University, entitled "Discussion and History of Certain Geometrical Problems of Heraclitus and Appollonius." He further says that the solution given there involves curves which are beyond the scope of "ruler and compass." He also refers to a solution by the hyperbola given in the *Educational Times Reprint*, Volume XX, New Series, page 37. The paper of Professor Archibald was read before the Edinburgh Mathematical Society on June 10, 1910.

My object in proposing the problem here is to show that it permits of a very simple solution by *Euclidean Geometry, i. e.*, by ruler and compasses only. The problem is very important and interesting, and deserves the attention of all geometricians.

NOTES AND NEWS.

The autumn meeting of the Southwestern Section of the American Mathematical Society was held at the University of Kansas on November 29 and 30. That of the San Francisco Section was held at the University of California on October 26. M.

On November 7, Editor Slaught delivered an address before the Iowa State Association of Teachers of Mathematics and Science, at their meeting in DesMoines, on the topic: "The changing conditions in relation to the secondary courses in mathematics." F.

The first number of volume 2 of the *Revista de la Sociedad Matematica Espanola,* which appeared in October, contains a new section devoted to bibliographic notices. Authors and publishers are invited to send copies of their publications for such notices to the Society at San Bernardo, 51, Madrid, Spain. M.

The nineteenth annual meeting of the American Mathematical Society will be held at Cleveland, Ohio, December 31 to January 2, in connection with the annual convocation of the American Association for the Advancement of Science. The winter meeting of the Chicago Section will be merged with the annual meeting at Cleveland. M.

The twelfth annual meeting of the Central Association of Science and Mathematics Teachers was held at Northwestern University, Evanston, Illinois, on Friday and Saturday, November 29 and 30. An important program for mathematics teachers occupied two sessions, on Friday afternoon and Saturday morning. S.

The Illinois Chapter of the Society of the Sigma Xi held its October meeting in honor of M. Henri Poincaré. Professor Borel of Paris spoke and called especial attention to the emphasis which his distinguished colleague had placed on the concepts of invariants and groups. Professor J. B. Shaw read a paper on Poincaré as an investigator. M.

Professor E. Borel, of Paris, France, and Professor Vito Volterra, of Rome, Italy, who were in this country for the purpose of taking part in the dedication of Rice Institute at Houston, Texas, made visits to several universities, including Illinois, Chicago, and Wisconsin in the West, and Columbia in the East. Professor Volterra also attended the meeting of the American Mathematical Society held in New York at the time of his visit there. S.

The Card Section of the Library of Congress expects to have two sets of cards for the great German and French Mathematical Encyclopedias ready for sale during December of the present year. The two sets are known as "author set" and "dictionary set." The latter contains enough copies of

the cards to make all the entries required in a dictionary catalogue and will probably cost about twice as much as the former set. Standing orders for these cards are invited at two cents for one copy of each card, and seven-tenths of a cent for each after the first entry. These cards will doubtless make the great mathematical encyclopedias very much more useful, especially since the regular indices to many of the volumes cannot appear for a number of years. M.

The readers of the MONTHLY are referred to a sketch of the life and works of the late Professor Henri Poincaré in *Science* for August 4, 1912, by Professor G. A. Miller. Professor Poincaré died suddenly on July 17. He had expected to attend the International Congress at Cambridge in August, and was to be one of the speakers at the dedication of Rice Institute at Houston, Texas, in October. Other noted mathematicians who gave addresses at this dedication were Professor Emile Borel, of the University of Paris, Professor Carl Störmer, of the University of Christiania, and Professor Vito Volterra, of the University of Rome. Professor Borel also gave lectures at the University of Illinois, the University of Chicago, and the University of Wisconsin.

Beginning in January, 1913, with Volume XX, THE AMERICAN MATHEMATICAL MONTHLY will pass under the control of an Editorial Board consisting of one representative from each of the nine institutions which have contributed toward a subsidy fund for its promotion, together with B. F. Finkel, the founder of the journal.

The contributing institutions are Colorado College, and the Universities of Chicago, Illinois, Missouri, Minnesota, Nebraska, Kansas, Indiana, and Iowa. The editorial representatives are respectively, Florian Cajori, H. E. Slaught, G. A. Miller, E. R. Hedrick, W. H. Bussey, W. C. Brencke, C. H. Ashton, R. D. Carmichael, and A. G. Smith.

It is proposed to make this journal appeal strongly to the great body of teachers of mathematics in the collegiate and advanced secondary fields, being careful not to encroach upon the domain of any existing journals, either above or below. The animating spirit of this reorganization is the conviction that such a journal is needed not only to direct attention to questions of improvement in teaching in these fields, but also to foster the development of the scientific spirit among large numbers who are not now reached by the more highly technical journals.

The Monthly will continue to publish carefully selected scientific articles, and will encourage the production of such articles by students and teachers in the earlier stages of research work, but relatively greater attention than heretofore will be given to pedagogical and historical questions of interest and value to teachers of collegiate mathematics. For instance, an important feature of the new volume beginning in January will be the serial

publication of Professor Cajori's latest research on "The History of the Logarithmic and Exponential Concepts."

Notwithstanding this proposed enlargement of the Monthly and the doubling of its cost of production, due to the employment of the best available service in mathematical printing, the subscription price will remain at Two Dollars per year. This is made possible only by the subsidy contributions mentioned above, and it is done with the hope that the subscription list may speedily grow to such proportions as to make the journal self-supporting.

Renewals and new subscriptions should be made payable to the Treasurer, B. F. Finkel, Springfield, Mo.

Contributed articles and official correspondence should be addressed to the Managing Editor, H. E. Slaught, 5548 Monroe Avenue, Chicago, Illinois, while all problems and solutions should be sent, for the present, to B. F. Finkel, Springfield, Mo.

The Monthly will soon issue an index of Volumes I-XIX, which will render accessible the large number and variety of contributions already published in this journal. Back numbers can be supplied at 25 cents each, and single volumes at two dollars and fifty cents each. No complete sets of the MONTHLY are available, many of the numbers being out of print. All correspondence relating to back numbers should be directed to B. F. Finkel. S.

Having gotten behind one number, we are obliged to combine two numbers in one that we may get the December number out on time and thus have the way clear for the January number, which is to come out under the new organization. F.

BOOKS.

The ABC of the Differential Calculus. By William Dyson Wansbrough, Author of the "Portable Steam Engine: Its Construction and Management;" The Proportions and Movement of Slide Valves;" "Modern Steam Boilers;" etc. Third Edition. 16mo. Cloth, xii+148 pages. Price, $1.50. New York: D. VanNostrand Co.

The author attempts to give the beginner an understanding of the Calculus by beginning with its most simple concepts. His illustrations are so simple and the details carried out so fully that any one ambitious enough to desire an insight into one of the most interesting of all mathematical subjects can readily gain such insight by perusing this book. F.

A Manual of Laboratory Exercises in Physics. By Frederick R. Gorton, B. S., M. A., Ph. D., Professor of Physics, Michigan State Normal College. 8vo. Cloth, xv+166 pages. New York: D. Appleton & Co.

In this little book is set forth a number of simple experiments in mechanics, sound, heat, light, and electricity and magnetism. F.

Differential and Integral Calculus. An Introductory Course for Colleges and Engineering Schools. By Lorrain S. Hulburt, Collegiate Professor of Mathematics in the Johns Hopkins University. 8vo. Cloth, xviii+481 pages. New York: Longmans, Green & Co.

This volume of the Calculus is divided into six books, as follows: Book I, treats of the differential calculus; Book II, of the integral calculus; Book III, introduction to analytical geometry of three dimensions, Book IV, functions of more than one argument; Book V, theorems of Taylor and MacLaurin, integration of rational fractions, and envelopes; Book VI, an introduction to ordinary differential equations.

The various topics of the Calculus discussed in this book are treated with great clearness and without undue complexity of notation; the problems have been carefully selected, some of them from the older texts and many of them have been created by the author himself. The publishers have been very fortunate in presenting the book in a very attractive style as to type and diagrams. F.

Daytime and Evening Exercises in Astronomy. For Schools and Colleges. By Sarah Francis Whiting, Sc. D., Whitin Observatory, Wellesley College. 8vo. Cloth, xiv+104 pages. New York, Boston, and Chicago: Ginn & Co.

It is hoped that this little book will stimulate an interest in the study of astronomy, and that we shall again find a large number of students in our colleges demanding courses in this oldest and one of the most inspiring of the sciences. F.

New Analytical Geometry. By Percy F. Smith, Professor of Mathematics in the Sheffield Scientific School of Yale University, and Arthur Sullivan Gale, Ph. D., Professor of Mathematics in the University of Rochester. 8vo. Cloth, x+342 pages. New York, Boston, and Chicago: Ginn & Co.

This book holds an intermediate place between the authors' *Introduction to Analytical Geometry* and the *Elements of Geometry.* The method of treatment is the same as that pursued in the two previous books. Those teachers who have found the *Elements* too exhaustive and the *Introduction* too limited will be glad to have this book, which will satisfy them in every particular. F.

Plane Geometry. By William Belz, A. M., Vice-Principal and Head of the Department of Mathematics in the East High School, Rochester, New York, and Harrison E. Webb, A. B., Head of the Department of Mathematics in the Central Commercial and Manual Training High School, Newark, New Jersey, with the editorial coöperation of Percy F. Smith, Professor of Mathematics in the Sheffield Scientific School of Yale University. 8vo. Cloth, x+332 pages. Price, $1.00. Boston, New York, and Chicago: Ginn & Co.

The authors' apology for adding another text to the long list of existing geometrical text-books is to effect a compromise between the extreme demands of certain ultra progressives and the equally untenable position of the stand-patters. To this end they have written this text with the following features: (1) A preliminary course followed by the demonstrative course; (2) The demonstrative course built up not only in a topical, but also in a psychological order; (3) Methods embodied in the text aiming to make pupils independent of the printed page; (4) Equal consideration to various types of exercises; (5) An extensive but not excessive list of applied problems; and (6) Providing a minor and a major course. A student who has the mastery of this text will be well equipped for the practical problems which he may meet in life. F.

Elements of Plane Trigonometry. High School Edition. By Robert E. Moritz, Ph. D. (Nebraska), Ph. N. D. (Strassburg), Professor of Mathematics, University of Washington. Small 8vo. Cloth, viii+315 pages. Price, $1.00. New York: John Wiley & Sons.

This book, we are told, is an effort on the part of the author to bring about a more perfect adjustment of the teaching of trigonometry with the teaching of the subjects on which it rests, and with the progress of the arts and sciences to which it applies. The book justifies the author's claim. F.

A History of the Theories of Aether and Electricity from the Age of DesCartes to the Close of the Nineteenth Century. By E. T. Whittaker, Hon. Sc. D. (Dublin); F. R. S.; Royal Astronomer of Ireland. New York and London, England: Longmans, Green & Co. 8vo. Cloth, xix+475 pages. Price, $3.00.

The scope of this splendid work may be inferred from its table of contents. The first chapter treats of the theory of the æther in the seventeenth century; chapter II, Electric and magnetic science, prior to the Introduction of the Potentials; chapter III, Galvanism, from Galvani to Ohm; chapter IV, The luminiferous medium, from Bradley to Fresnel; chapter V, The æther as an elastic solid; chapter VI, Faraday; chapter VII, The mathematical electricians of the middle of the nineteenth century; chapter VIII, Maxwell; chapter IX, Models of the æther; chapter X, The followers of Maxwell; chapter XI, Conduction in solutions and gases, from Faraday to J. J. Thompson; chapter XII, The theory of æther and electrons in the closing years of the nineteenth century.

In the treatment of some of the subjects, vector analysis is used. To make the book readable to those unfamiliar with vector analysis, the author has introduced at the beginning a page of vector notation. This work will be of inestimable value to the teacher of physics. It was prepared by an able and eminent writer on physical and mathematical subjects, a fact insuring reliability and accuracy in statement and discussion. F.

Laboratory Studies in Chemistry. By Robert H. Bradbury, A. M., Ph. D., Head of the Department of Science in the Southern High School, Philadelphia. 8vo. Cloth, ix+129 pages. New York and Chicago: D. Appleton & Co.

This manual covers the various syllabi which teachers preparing students for college have to consider, and requires simple apparatus and generally inexpensive material. F.

A Shorter Geometry. By C. Godfrey, M. V. O., M. A., Head Master of the Royal Naval College, Osborne; formerly Senior Mathematical Master at Winchester College; and A. W. Siddons, M. A., late Fellow of Jesus College, Cambridge; Assistant Master at Harrow School. 8vo. Cloth, xxii+301 pages. Price, 80 cents. Cambridge: The University Press. G. P. Putnam's Sons, American Agents.

The plan of this book presents the subject in three stages, as follows: First, introductory work concerned with fundamental concepts; Second, discovery of fundamental facts of geometry, by experiment and intuition; and, Third, subsequent deductive development of the propositions. The book is beautifully printed on good paper, and handsomely bound. The authors are neither ultra-radical nor ultra-conservative. They clearly recognize the weakness of many of the modern methods as well as those of former times. The book is pedagogically wholesome. F.

Mathematical Wrinkles. For Teachers and Private Learners. Consisting of Knotty Problems; Mathematical Recreations, Answers and Solutions; Rules of Mensuration; Short Methods; Helps, Tables, Etc. By Sam I. Jones, Professor of Mathematics in the Gunter Biblical and Literary College, Gunter, Texas. 12mo. Half Leather, viii+321 pages. Price, $1.65.

This book ought to be in the library of every teacher who has to teach any of the mathematical subjects in high schools, academies, or country schools. It contains a vast amount of useful and interesting material which such teachers ought to know, and with which they can often arouse interest and stimulate enthusiasm in their classes. It is a very regrettable fact that the larger per cent of our teachers who teach mathematics have little vital interest in the subject, and this fact is manifested by the extreme poverty of their libraries, not only in mathematics but in other subjects as well. Teachers who have not enough interest in their subjects to keep themselves informed on what is going on in this rapidly progressing world, had better give their places to others who have a higher ambition than merely making a living. Let the teacher of elementary mathematics buy a copy of this book. It will do you good. It contains problems, solutions, quotations from mathematicians, puzzles, etc. F.

Practical Descriptive Geometry. By William Griswold Smith, M. E., Assistant Professor of Descriptive Geometry and Kinematics, Armour Institute of Technology. 8vo. Cloth, ix+208 pages. New York: McGraw-Hill Book Company.

The author of this book, while recognizing the indisputable excellence of many of the text-books on Descriptive Geometry, feels that many of these excellent treatises are only useful as reference books; others are incomplete in essentials; still others are faddish, emphasizing certain features and treating others inadequately; while even the best convey only a slight idea of the practical value of the subject. Keeping these defects before him, we believe he has written a book that is accurate, practical, and teachable. F.

Essentials of Calculus. By E. J. Townsend, Ph. D. (Göttingen), Professor of Mathematics, University of Illinois, and G. A. Goodenough, M. E. (Illinois), Associate Professor of Mechanical Engineering, Univsrsity of Illinois. 8vo. Cloth, x+355 pages. Chicago: Henry Holt & Co.

The average college student and students of technical schools will find their needs quite well met in this book. The work is based on the theory of limits, and the usual division of the subject into differential and integral has been pretty generally disregarded. This text will meet the demands of a large number of teachers in our colleges and technical schools. F.

Gravitation. By Frank Harris, B. A. (Oxon), Late Executive Engineer, and Associate M. Inst. C. E. 8vo., xi+107 pages. Price, $1.00. New York: Longmans, Green & Co.

The author presents, in this volume, a theory as to the nature of gravitation. He begins his discussion with an inquiry concerning the motion of a massless spherical shell in an incompressible, frictionless fluid of unit density. He next considers the motion of two spheres in an infinite fluid. His discussion, in which he uses mathematical reasoning freely, is interesting and instructive. F.

Complete School Algebra. By Herbert E. Hawkes, Ph. D., Professor of Mathematics in Columbia University; William A. Luby, A. B., Head of the Department of Mathematics, Central High School, Kansas City, Mo.,

and Frank C. Tuton, Ph. B., Principal of Central High School, St. Joseph, Mo. 12mo Cloth, x+507 pages. Price, $1.25. Boston and Chicago: Ginn & Co.

"The 'Complete School Algebra,' which includes between the covers of a tingle volvme—with the necessary adaptation and abridgment—all the material of the authors' 'First Course in Algebra' and 'Second Course in Algebra,' is designed for those schools which find a one-book course best suited to their needs.

"The first twenty-three chapters contain the greater portion of the work usually taken up during the first year. Then follows the review material, each topic being given a broader and more advanced treatment than is permissible in first year work. New matter is used throughout, and many new applications are given in order to make a fresh and inviting appeal to the student. In the remaining chapters those advanced topics considered necessary by the best secondary schools are included.

The Hawkes, Luby, and Touton Algebras are marked by freshness and sanity of method. Wealth of illustrative material, correlation with arithmetic, geometry, and physics, prominence given the equation, emphasis on checking, and extensive work with graphs are a few of the features. These algebras—in a one-book and a two-book series—offer an arrangement of ideal flexibility in their adaptation to the needs of different schools."

Lectures on the Theory of Functions of Real Variables. Volume II. By James Pierpont, LL. D., Professor of Mathematics in Yale University. 8vo. Cloth, xiii+645 pages. Price, $5.00.

In the compilation of this second volume, the author has rendered the American mathematician an invaluable service and has added to American Mathematical literature a most creditable and monumental work

The present volume comprises seventeen chapters treating the following subjects in order: Point Sets and Proper Integrals; Improper Multiple Integrals; Series; Multiple Series; Series of Functions; Power Series; Infinite Products; Aggregates; Ordinal Numbers; Point Sets, Measure; Lebesgue Integrals; Fourier's Series; Discontinuous Functions; Derivatives, Extremes, Variation; Sub- and Infra-Uniform Convergence; and Geometric Notions.

Pierpont's two volumes together with Hobson's treatise on the same subject present a fairly complete discussion of all that is known on the Theory of Functions of a Real Variable at the present time.

The publishers are to be congratulated for the very excellent style of printing and binding adopted in this work. F.

Syllabus of Mathematics. A Symposium compiled on the Teaching of Mathematics to Students of Engineering. Accepted by the Society for the Promotion of Engineering Education at the Nineteenth Annual Meeting held at Pittsburgh, Pennsylvania, June, 1911. 8vo. Cloth, 136 pages. Price, 75 cents. Ithaca: Office of the Secretary.

The Society feels that this report should be of value to teachers of mathematics in showing them what are considered fuudamentals for engineering subjects and to indicate to teachers of engineering the preparation which they may reasonably expect their students to have had. It should also serve as a reference syllabus for students or engineers who wish to systematize their knowledge of mathematics. F.

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RETROSPECT AND PROSPECT.

By H. E. SLAUGHT, The University of Chicago.

With this issue THE AMERICAN MATHEMATICAL MONTHLY closes the nineteenth year of its history. It was founded in 1894 by Benjamin F. Finkel, then professor of mathematics at Kidder Institute, Kidder, Missouri. Associated with him in the editorship was J. M. Colaw; then principal of the high school at Monterey, Virginia. Other associate editors who have assisted from time to time are Saul Epsteen, during the years 1904-5 and Oliver E. Glenn during 1906.

Professor Finkel, as author of the well known Mathematical Solution Book, naturally gave great impetus, through the columns of his newly established journal to the proposal and solution of problems. The number and character of the contributors to these departments and the sustained interest in these contributions even up to the present time, on the part of the readers of this journal are of no small import in its history. But appropriate space has also been given to contributed articles of a widely diversified character, including biographies of mathematicians, historical, pedagogical, and bibliographical papers, and elementary research in pure and applied mathematics. The names of the contributors constitute a long and worthy list, as will be evident from the general index to volumes I to XIX soon to be published. These contributions have not only provided interest and profit for the constituency of the Monthly, but, in not a few cases, they have been veritable stepping stones on which the authors themselves have risen to greater mastery of technical research in the more advanced phases of mathematics.

It is in nowise the purpose of this article to relate the history of the Monthly in detail, in fact this could hardly be done with completeness except by the founder himself and those who were intimately associated with him during the early years. Suffice it to say that the years 1894-1902 constitute an epoch in the history during which the full financial and editorial responsibility rested upon the founder alone, while in the period 1903-1912, the full editorial responsibility, aside from the problem departments, rested upon others, and some financial assistance was received from outside sources.

In January 1903, Dr. Leonard E. Dickson, then assistant professor of mathematics at the University of Chicago, became co-editor and assumed full responsibility for the acceptance and publication of contributed articles, Editor Finkel retaining control of the problem departments and business management. This arrangement was exceedingly fortunate, not only because of Professor Dickson's eminent fitness for this service but because of his loyal and continuous interest in the Journal from its inception, having contributed many papers to its columns, including an article for the first number of volume one, while he was a student of the University of Texas.

Prof. Dickson continued in this relationship until January, 1907, when on account of pressing duties in other directions he succeeded in inducing the present writer to take his place. During Professor Dickson's editorship the quality of the journal was maintained on a high standard, but, in common with most scientific journals, its subscription receipts were not sufficient to cover the cost of publication. In fact, if all the history were told, it would be found that Editor Finkel, as owner and business manager had more than once personally made up deficits previous to this time, and for such devotion and unselfish contributions to the support of this journal he deserves and should receive the gratitude of all who believe in the mission of such a publication in a field otherwise quite unoccupied in this country. Professor Dickson believed in the mission of the MONTHLY and he saw and appreciated this service begun and continued by its founder, and, upon representation to the University of Chicago of the facts in the case, he succeeded in securing from this source a small annual subsidy, which has been continued to the present time.

The writer also believed in the mission of the MONTHLY and knew that it deserved more friends and supporters. Accordingly he at once began a campaign of correspondence with men the country over who were in positions of influence and whose assistance might be secured both in strengthening the status of the MONTHLY and in providing further subsidy funds. The proposition was put forth, in season and out of season, that a journal is needed in this country which should give more direct attention to the teaching of mathematics in the colleges, that should not only provide articles of a scientific nature whose technical difficulties are within the comprehension and control of the average college teacher, but should also deal frankly and scientifically with historical and pedagogical questions which concern the mathematical courses given in the more advanced secondary schools and the undergraduate college curricula. The responses for the most part indicated static interest rather than dynamic, the feeling being that any important advance with respect to the MONTHLY must involve at the outset its removal to some metropolitan center where more ample facilities could be secured for its printing and publication and for this would be needed a large increase in income which was not in sight.

But presently Professor Townsend of the University of Illinois lent

a willing ear and through his influence the University of Illinois joined the University of Chicago in lending financial support to the MONTHLY and Professor G. A. Miller became associated with the writer in joint control of contributed articles, Editor Finkel remaining as before in charge of the problem department and business manager. This arrangement began in January, 1909, and has continued to the present. This action on the part of the University of Illinois and the unselfish devotion and effective co-operation of Professor Miller may well be counted as the turning point in the history of the movement which has now culminated in the formal transfer of the ownership and control of this journal to a board of ten editors, including the founder, and nine representatives of institutions which have agreed to contribute to the subsidy fund in order to put the new organization on its feet.

Without this action on the part of the University of Illinois at that critical time, the writer is free to say that he would soon have relinquished further effort and retired from the field. With this encouragement, however, he renewed his efforts to enlist active interest and co-operation. The next response came from Colorado College through the influence of Professor Florian Cajori, and the next from the University of Missouri through the influence of Professor E. R. Hedrick. Then followed the Universities of Minnesota, Nebraska, Kansas, Indiana, and Iowa, with Professors Bussey, Brenke, Ashton, Carmichael and Baker as editorial representatives.

This Board of Editors is organized under articles of agreement whereby it is self-perpetuating and all matters of editorial and business policy are determined by majority vote of the members, including the election of officers and committees. Some phases of the proposed editorial policy will be set forth in a "Foreword" on behalf of the editors in the January, 1913, issue, which will begin the new regime. But the most vital consideration is that the editorial policy shall be a living and growing thing, a conviction that important service is to be rendered and that the best attainable means that are to be used.

The plan involves a greatly increased expenditure and this must be met by a corresponding increase in income. It might have been done by doubling the subscription price, which, under present cost for material and service, would not have been unreasonable. But it was decided to keep the subscription price at two dollars per year in the hope that the list of subscribers may be speedily doubled, yes twice doubled, in which case the journal can become self-supporting. Meanwhile, the subsidy fund, supplemented by some selected advertising along mathematical lines, will put the new organization in motion.

It is urged that all subscribers who are in arrears come to the rescue of the present management in closing up its accounts, and that all renewals be made in *advance* to the end that the new management may be able to meet its contracts with promptness.

All subscriptions should be made payable to THE AMERICAN MATHEMATICAL MONTHLY and sent directly to the Treasurer, B. F. Finkel, Springfield, Missouri. Foreign subscriptions should include fifty cents extra to cover postage for foreign delivery.

All articles contributed for publication should be sent to the Managing Editor, H. E. Slaught, 5548 Monroe Avenue, Chicago, Illinois, or to one of of the other members of the Editorial Committee: namely, G. A. Miller, University of Illinois, Urbana, Illinois, or to E. R. Hedrick, University of Missouri, Columbia, Missouri.

All business correspondence should be addressed to the Managing Editor.

Problems for solution should be sent to the chairman of this committee, B. F. Finkel, or to one of the other members; namely, to R. P. Baker, University of Iowa, Iowa City, Iowa, or to W. C. Brenke, University of Nebraska, Lincoln, Nebraska.

Books for review should be sent by publishers or authors to W. H. Bussey, University of Minnesota, Minneapolis, Minnesota, chairman of this committee.

News items, such as appointments, promotions, notice of meetings, biographical and bibliographical notes, historical and pedagogical references, etc., should be sent to G. A. Miller, University of Illinois, Urbana, Illinois, chairman of this committee.

The MONTHLY depends upon its present subscribers and loyal supporters to aid in spreading its influence and especially to assist in swelling its subscription list. It is important that new subscribers should begin with the January, 1913, issue, especially because this will contain the first instalment of Professor Cajori's latest research on "The History of the Logarithmic and Exponential Concepts," which is to be published serially during the coming months.

PARALLELOGRAMS INSCRIBED IN A RECTANGLE.

By L. A. HOWLAND, Wesleyan University, Middletown, Connecticut.

In the May, 1912, number of the MONTHLY, Professor Hodge points out the desirability of having a stock of a certain type of problems for use in a course in coördinate geometry. There is another type which seems to me to be also desirable for such a course. Not infrequently we find a student who has originality and is willing to think. His ability is not used to the full in the establishment of prescribed results. If started on the proper kind of problem, he is capable of discovering, what are to him at least, new results. The following seems to be such a problem, because its solution and

discussion suggest a large number of theorems, some of them by no means obvious or well known, yet discoverable by very elementary methods.

> *Problem:** Given a rectangle R of length $2a$ and breadth $2b$, to inscribe in it a parallelogram P, whose shorter side is h and whose angle with vertex on longer side of R is arc tan m.

We will take two sides of R as coördinate axes (Fig. 1), and let the coördinates of the vertices A and B be $(0, Y)$ and $(X, 0)$, respectively. Those of D and C will be $(2a-X, 2b)$ and $(2a, 2b-Y)$, respectively.

We have $X^2+Y^2=h^2$.

The equation of BC is $y=\frac{2b-Y}{2a-X}(x-X)$.

The equation of AB is $y=-\frac{Y}{X}(x-X)$.

If $\theta=\angle ABC=$arc tan m, we have

$$\tan\theta=-\frac{2bX-2XY+2aY}{2aX-X^2-2bY+Y^2}=m.$$

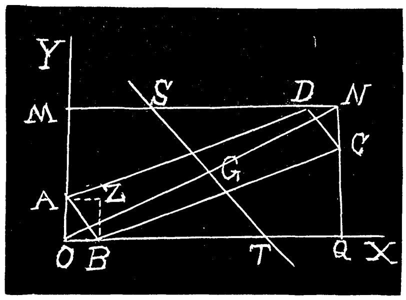


Fig. 1.

It appears then that X, Y must be solutions of the simultaneous equations

$$x^2+y^2=h^2\ldots(1),$$
$$mx^2+2xy-my^2-2(b+am)x-2(a-bm)y=0\ldots(2).$$

In other words, *the point Z is determined as the intersection of the circle* (1) *and the hyperbola* (2). From the position of Z we have at once the position of A and B and hence of C and D, and $ZN=l$, the length of P.

The hyperbola (2) is equilateral. Its center is the point (a, b), the center G of R. Its principal axis makes an angle $\frac{1}{2}$ arc tan$\frac{1}{m}$ with the X-axis. Its semi-axes are

$$\sqrt{\frac{m(a^2-b^2)+2ab}{\sqrt{(1+m^2)}}}.$$

Hence the vertices of these hyperbolas lie on the curve

$$r=\sqrt{\frac{m(a^2-b^2)+2ab}{\sqrt{(1+m^2)}}},$$

* Suggested by the example in Osgood's *Calculus*, page 404.

$$\theta=\tfrac{1}{2}\operatorname{arc}\tan\frac{1}{m},$$

referred to G as pole and a parallel to the X-axis as initial line. Eliminating the parameter m, we have

$$r^2=(a^2-b^2)\cos 2\theta+2ab\sin 2\theta.$$

If we take the diagonal of R as the initial line, we have $\theta=\phi-\operatorname{arc}\tan\frac{b}{a}$, and the equation of the curve becomes

$$r^2=(a^2+b^2)\cos 2\phi.$$

This is a lemniscate with center at G and vertices at the vertices O and N of R.

The slope of the hyperbola at the origin is

$$s=\frac{b+am}{bm-a}.$$

Since the hyperbola lies on the opposite side of the tangent from its center, the branch of the curve through O (and, owing to symmetry, the other branch also) cannot cut into R, and hence there can be no P, unless $0<s<\infty$. That is, unless

$$\begin{cases} b+am>0 \\ bm-a>0 \end{cases} \text{ or } \begin{cases} b+am<0 \\ bm-a<0 \end{cases}$$

This gives $m>\frac{a}{b}$ or $m<-\frac{b}{a}$.

These conditions may be expressed as follows: If $\alpha=\operatorname{arc}\tan\frac{a}{b}$, then must $\angle ABC$ lie between α and $\alpha+90°$. *The extent of the range of possible values for* $\angle ABC$ *is then always* 90° *entirely independent of the dimensions of* R.

For $m=-\frac{2ab}{a^2-b^2}$, the hyperbola (2) becomes

$$(bx-ay)(ax+by-a^2-b^2)=0,$$

i. e., it degenerates into the diagonal ON and its perpendicular bisector ST. It can be shown that ST cuts none of the hyperbolas (2),—itself excepted,—in real points.

It is evident that when an h circle intersects the line ST,—and only then,—an inscribed rhombus will be determined. It is also evident that the maximum value of h is $OT=OS=\frac{a^2+b^2}{a}$.

A consideration of these results will suggest the following properties or theorems:

1. There can be no P inscribed in R unless $m>\frac{a}{b}$ or $m<-\frac{b}{a}$, *i. e.*, unless angle ABC lies between α and $\alpha+90°$, where $\alpha=\text{arc tan}\frac{a}{b}$.

2. There can be no P inscribed in R unless $h \leqq \frac{a^2+b^2}{a}$.

3. For a given $h<\frac{a^2+b^2}{a}$ there will be an infinite number of P's, whose m's fill a definite interval, m_1 to m_2, dependent upon the size of h,

Problem: To determine this interval.

4. For a given m, satisfying the conditions of No. 1, there will be an infinite number of P's, whose h's vary from zero to a certain maximum value H, dependent upon the size of m.

Problem: To determine H.

5. For a given m, satisfying the conditions of No. 1, and a given h, less than or equal to the corresponding H, there will be one, and only one, P.

6. For a given m there may be two P's of the same length but of different h, or for a given h there may be two P's of the same length but of different m. In neither case can there be more than two.

Problem: To determine when two are possible.

7. In R there can be inscribed an infinite number of rhombuses, whose sides vary in length from $\sqrt{(a^2+b^2)}$ to $\frac{a^2+b^2}{a}$ inclusive. They all, however, have the same angles, $\angle ABC$ being $\arctan\left(-\frac{2ab}{a^2-b^2}\right)$.

8. No square can be inscribed in R unless R is itself a square.

9. In a square of side $2a$ there can be inscribed an infinite number of rectangles of breadth varying from 0 up to and including 2a. Of these, those whose breadth lies between $a\sqrt{2}$ and $2a$, both limits included, will be squares.

10. No rhombus can be inscribed in a square.

In the special case where a rectangle is to be inscribed in R, the hyperbola is

$$x^2-y^2-2ax+2by=0.$$

Its axis is parallel to the X-axis and its semi-axis is $\sqrt{(a^2-b^2)}$. Since its eccentricity is $\sqrt{2}$, the distance of its foci from the center is $\sqrt{[2(a^2-b^2)]}$, and that of its directrices is $\frac{1}{2}\sqrt{[2(a^2-b^2)]}$. It may be neatly constructed as follows (Fig. 2):

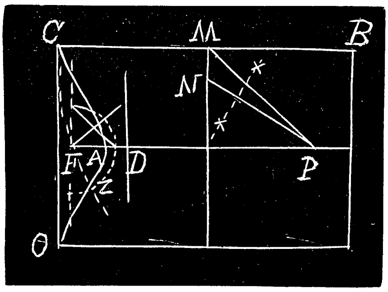


Fig. 2.

Lay off from M the distance $MP=a$. Then $O'P=\sqrt{(a^2-b^2)}$ is the distance from O' to the vertex. Lay off $O'N=O'P$. Then $NP=\sqrt{[2(a^2-b^2)]}$ is the distance from O′ to the focus F, and $NQ=\frac{1}{2}NP$ is the distance from O′ to the directrix. Draw a circle of any radius about F. Draw the chord of a quadrant and bisect it. The points of intersection of the circle and a line parallel to the directrix at a distance from it equal to the half chord lie on the hyperbola.

The applications of this problem are, of course, well known. For the sake of definiteness let us take the problem of finding the longest rectangle of a given width which can be laid on the floor of a rectangular room. If in Fig. 2 we draw a circle of radius $2a$ about B, it will cut the hyperbola at C and again at a point Z_1. If h is equal to or greater than OZ_1, BZ will obviously be equal to or less than $2a$. Hence if the rectangle has a width between OZ_1 and $2a$, it should be laid lengthwise of the room and can have a length $2a$. If its width is less than OZ_1, it should be "inscribed." Its length, BZ, will be greater than $2a$.

Fig. 2 is constructed for a rectangle 12×16 cm. It will obviously serve for any similar rectangle by proper choice of scale. OZ_1 measures 4.4+ cm.* Hence on the floor of a room $12'\times16'$ rectangles of width approximately 4′.4 or wider can be but 16′ long. Narrower rectangles can be longer. Measurement gives the length of one 3′ wide to be 17.2′+.*

NOTE ON LAMBERT'S METHOD OF SOLVING LINEAR DIFFERENTIAL EQUATIONS.

By H. L. SLOBIN, University of Minnesota.

In the July, 1910, *Annals*, Professor Lambert gives Cauchy's Method of Solving Linear Differential Equations, by breaking up the function

$$f(x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \ldots,\ \frac{d^ny}{dx^n})=0\ldots(1),$$

* Actual computation gives these values to four places of decimals to be: $OZ_1=4.4165$ and $BZ=17.2037$.

into *two* parts, f_1 and f_2, and introducing a parameter t so that $f=0$ is replaced by $f_1+tf_2=0\ldots(2)$.

He then assumes as a solution of (2),

$$y=A+Bt+Ct^2+\ldots, \ldots(3),$$

where A, B, C, ... are undeterminate functions of x, and substituting the expression (3) for y in (2) he equates to zero the coefficients of the successive powers of t, and solves the *differential equations* thus obtained for A, B, C, ...

Substituting these values obtained for A, B, C, ... in (3), and replacing it by unity, a series for y results, which, if convergent, is a solution of the differential equation.

It might be worthy of note that since the process is purely formal, and since t is arbitrarily introduced, in the end to be replaced by unity, we may break up the function into *several parts*, and introduce *various powers of t*, and then proceeding exactly as stated in Lambert's paper, the work for determining A, B, ... may frequently be much simplified; and where only *particular* solutions are desired, *the differential equations that must be solved to determine A, B, ... may be of lower degree* than the original equation. The aim of course is to obtain A, B, C, ... in the simplest form and with least difficulty.

Each particular example will readily suggest what powers of t should be introduced with the various parts into which f was broken up.

I illustrate my suggestion by applying it to the two examples used by Professor Lambert.

I. *For general solution:*

$$\frac{d^2y}{dx^2}+ax^2y=1+x\ldots(1).$$

$$\frac{d^2y}{dx^2}-1.t^2-xt^3+ax^2yt^4=0\ldots(2).$$

$$y=A+Bt+Ct^2+\ldots \quad \ldots(3).$$

$$\left.\frac{d^2A}{dx^2}\right| + \left.\frac{d^2B}{dx^2}\right| t + \left.\begin{matrix}\dfrac{d^2C}{dx^2}\\ -1\end{matrix}\right| t^2 + \left.\begin{matrix}\dfrac{d^2D}{dx^2}\\ -x\end{matrix}\right| t^3 + \left.\begin{matrix}\dfrac{d^2E}{dx^2}\\ +ax^2A\end{matrix}\right| \begin{matrix}t^4+\ldots\\ +\ldots\end{matrix} \equiv 0.$$

Equating to zero the coefficients of the successive powers of t, using

only particular solutions in every case, since we only need two independent constants in the general solution, we have immediately, by inspection,

$$A=c_1,\ \ B=c_2x,\ \ C=\frac{x^2}{2\,!},\ D=\frac{x^3}{2.3},\ E=-aC_1\frac{x^4}{3.4},\ \text{etc.}$$

where the individual terms are exactly the same as in Lambert's solution, but where A, B, ... are monomials. Lambert shows that the series for y thus obtained is the sum of four convergent series, and that $y=A+B+C+$ ... is a solution.

II. *For a particular solution:*

$$x^2\frac{d^2y}{dx^2}+(x+2x^2)\frac{dy}{dx}-4y=0\ldots(1),$$

$$2x^2\frac{dy}{dx}+\left(x\frac{dy}{dx}-4y\right)t+x^2\frac{d^2y}{dx^2}t^3=0\ldots(2),$$

$$y=A+Bt+Ct^2+\ldots\ \ldots(3).$$

$$\begin{array}{l|l|l|l|l|l}
2x^2\dfrac{dA}{dx} & +2x^2\dfrac{dB}{dx} & t+2x^2\dfrac{dC}{dx} & t^2+2x^2\dfrac{dD}{dx} & t^3+2x^2\dfrac{dE}{dx} & t^4+\ldots \\
 & +x\dfrac{dA}{dx} & +x\dfrac{dB}{dx} & +x\dfrac{dC}{dx} & +x\dfrac{dD}{dx} & \equiv 0. \\
 & -4A & -4B & -4C & -4D & \\
 & & & +x^2\dfrac{d^2A}{dx^2} & +x^2\dfrac{d^2B}{dx^2} & +\ldots
\end{array}$$

Hence, $A=c_1$, $B=-2c_1x^{-1}$, $C=\frac{5}{2}c_1x^{-2}$, $D=-\frac{5}{2}c_1x^{-3}$, ..., etc., and series for y is obtained, which, if convergent, is a particular solution.

When by a different grouping of terms another independent particular solution can be obtained, the sum of the two will be the general solution. Of course, if the *general* solution is to be determined *directly*, the differential equations determining A, B, ... can not be of lower degree than the original differential equation, but as Lambert points out they are very much simpler than the given differential equation.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

377. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Expand in series $(1+x)^{1/x}$.

Solution by the PROPOSER.

If $y=(1+x)^{1/x}$, $\log y=\dfrac{\log\ (1+x)}{x}$.

$\log y=1-x/2+x^2/3-x^3/4+x^4/5-\ldots$

$\therefore y=e^{1-x/2+x^2/3-x^3/4+x^4/5-\ldots}=e\left(e^{-x/2+x^2/3-x^3/4+x^4/5-\ldots}\right)$

$$=e[1+(-x/2+x^2/3-x^3/4+\ldots+\frac{1}{1.2}(-x/2+x^2/3-x^3/4+\ldots)^2$$

$$+\frac{1}{1.2.3}(-x/2+x^2/3-x^3/4+\ldots)^3+\ldots]$$

$$=e(1-x/2+\tfrac{11}{24}x^2-\tfrac{7}{16}x^3+\tfrac{2447}{5760}x^4-\ldots)$$

Also solved by E. M. Harding, H. C. Feemster, and A. H. Holmes.

378. Proposed by C. N. SCHMALL, New York City.

Given $\sin^{-1}\dfrac{x}{a}+\sin^{-1}\dfrac{x}{b}+\sin^{-1}\dfrac{x}{c}=\pi$, solve for x.

Solution by E. B. ESCOTT, Ann Arbor, Michigan.

Let $\sin^{-1}\dfrac{x}{a}=\alpha$, $\sin^{-1}\dfrac{x}{b}=\beta$, and $\sin^{-1}\dfrac{x}{c}=\gamma$; whence $x=a\sin\alpha=b\sin\beta$ $=c\sin\gamma$.

$\dfrac{a}{b}=\dfrac{\sin\beta}{\sin\alpha}$; $\dfrac{a}{c}=\dfrac{\sin\gamma}{\sin\alpha}$; or $\sin\alpha:\sin\beta:\sin\gamma=\dfrac{1}{a}:\dfrac{1}{b}:\dfrac{1}{c}$.

Therefore, $1/a$, $1/b$, $1/c$ are the sides of a triangle whose corresponding angles are α, β, γ.

By the well-known formula of trigonometry:

$$\text{Area}=\tfrac{1}{2}.1/b.1/c.\sin\alpha=\sqrt{[s(s-1/a)(s-1/b)(s-1/c)]},$$
where $s=\tfrac{1}{2}(1/a+1/b+1/c)$;

whence, $\sin\alpha = 2bc\sqrt{[s(s-1/a)(s-1/b)(s-1/c)]}$;

$$x = a\sin\alpha = 2abc\sqrt{[s(s-1/a)(s-1/b)(s-1/c)]}$$

$$=\frac{1}{2abc}\sqrt{[(bc+ac+ab)(-bc+ac+ab)(bc-ac+ab)(bc+ac-ab)]}.$$

Solved similarly by W. S. Risley, H. C. Feemster, A. M. Harding, G. W, Hartwell, Elmer Schuyler, and A. H. Holmes.

GEOMETRY.

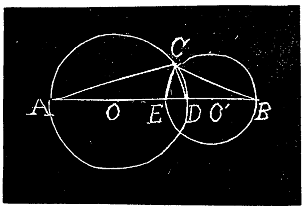
403. Proposed by C. E. GITHENS, Wheeling, West Virginia.

In a triangular field the sides enclosing an obtuse angle are 35 rods and 48 rods in length, Two straight lines are drawn from this vertex, and are at right angles to these sides. If these lines intersect the base 16 rods apart, how long is the third side of the field?

Solution by J. SCHETFER, A. M., Hagerstown, Maryland.

Let there be two intersecting circles; AB the line of centers, C one point of intersection. Draw AC, BC, CD, CE; put $AC=b$, $BC=a$, $DE=m$, radius of circles O and O', respectively, R and r.

In triangles ACE and BDC, respectively, we have $2R-m : b = \cos(A+B) : \cos B$; $2r-m : a=\cos(A+B) : \cos A$.



Eliminating $\cos(A+B)$ and considering $2R\cos A=b$, $2r\cos B=a$, we get $a^2R(2R-m)=b^2r(2r-m)$ .. (1).

Since $\sin A:\sin B=a:b$, we have $b^2(1-\cos^2 A)=a^2(1-\cos^2 B)$; therefore, $b^2r^2(4R^2-b^2)=a^2R^2(4r^2-a^2)$.

From (2) we get

$r^2=\dfrac{a^4R^2}{4R^2(a^2-b^2)+b^4}$...(3). Substituting in (1) and making all necessary reductions, involving merely simple algebraic operations, we get

$$16(a^2-b^2)(R^5-16m(a^2-b^2)R^4-4(a^2-b^2)(2b^2-m^2)R^3+4a^2b^2mR^2$$
$$+b^4(a^2-b\ \ -2m^2)R-b^6m=0,$$

or, divided by $16(a^2-b\ \ )$,

$$R^5-mR^4-\frac{2b^2-m^2}{4}R^3+\frac{a^{\cdot}b^2m}{4(a^2-b^2)}R^2+\frac{b^4(a^2-b^2-2m^2}{16(a^2-b^2)}R-\frac{b^6m}{16(a^2-b^2)}=0.$$

This equation of the fifth degree is to be solved for R, then r is found from (3), and $AB=2(R+r)-m$. For $a=35$, $b=48$, $m=16$, the equation is

$$R^5-16R^4-1088R^3-10463.024R^2+488494.5467R+6415370.2=0.$$

404. Proposed by S. LEFSCHETZ, Ph. D., University of Nebraska.

Let ABC be a triangle, M a point on BC, BE the intersection of the perpendicular to AM in M with AB and AC, F the other intersection of the circle circumscribed to ABC with the circle through M and A orthogonal to it in A. Prove that the points C, D, E, F are on a circle, and find the envelope of the latter when M describes BC.

No solution of this problem has been received.

405. Proposed by C. N. SCHMALL, New York City.

A plane cuts a constant volume from a given right cone. Prove that the minor axis of the section has a constant length.

Solution by the PROPOSER.

Let C be the vertex of the cone, AB the major axis of any one of the sections made by the plane. Let F be the point of contact of the inscribed circle of triangle ABC with the side AB. Then F is a focus of the section. Draw CD perpendicular to AB. Let V denote the volume cut off. Then, if α, β, denote the semi-axes of the section, we have

$$V=\tfrac{1}{3}CD.\,(\text{area of section})=\tfrac{1}{3}CD.\pi\alpha\beta=\tfrac{1}{3}\pi.CD.\alpha\beta$$
$$=\tfrac{1}{6}\pi.CD.AB.\beta=\tfrac{1}{6}\pi\, ab\sin C.\beta\ldots(1),$$

(where a, b, are the sides AC, BC, of the triangle ABC). But, $\beta^2=AF.FB=(s-a)(s-b)=ab\sin^2\tfrac{1}{2}C$. $\therefore ab=\beta^2\cos^2\tfrac{1}{2}C$.

Substituting this in (1), we get, $V=\tfrac{1}{3}\pi\beta^3.\tan\tfrac{1}{2}C$.

Now, since C and V are constant, therefore β, or the minor axis (2β) is also constant. It is now also evident, by (1), that the area of triangle ABC is constant.

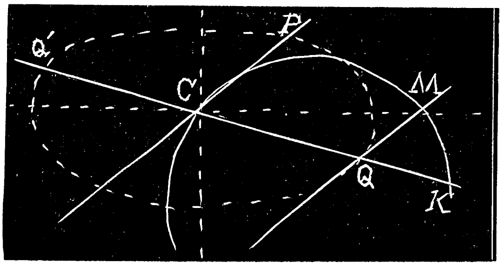
406. Proposed by DR. R. K. MORLEY, University of Illinois.

Given the lengths of a pair of conjugate diameters of an ellipse and the angle between them; to construct (with ruler and compass) the axes of the ellipse, *i. e.*, find their lengths and the angles they make with the given diameters.

I. Solution by A. M. HARDING, University of Arkansas.

Let PP' and QQ' be the conjugate diameters, and C the center. Produce CQ to K, so that $CQ.QK=CP$.

Draw a line through Q parallel to CP. Construct a circle with its center on this line and passing through C and K. Let the circle cut the line through Q at M and N. Then the axes of the ellipse will lie along CM and CN.



Proof. MN is tangent to the conic.

Now it can be easily shown that if a tangent meets any two lines in such a way that the product of its segments equals the square on the parallel semi-diameter, then the two lines are parallel to a pair of conjugate diameters. In the above construction, $NQ.QM=CQ.QK=CP^2$.

Therefore, CM and CN are the axes (since they are at right angles). Now let r=side of a square whose area is four times the triangle QCP; then $r^2=2ab$. Let s=hypotenuse of a right triangle whose legs are CP and CQ; then $s^2=CP^2+CQ^2=a^2+b^2$. Now construct $m=\sqrt{(s^2+r^2)}$ and $n=\sqrt{(s^2-r^2)}$; then $m=a+b$, $n=a-b$; hence, $a=\dfrac{m+n}{2}$, $b=\dfrac{m-n}{2}$.

II. Solution by H. C. FEEMSTER, York College, York, Nebraska.

Let $a'a$ and $b'b$ be the conjugate diameters, making the angle aOb. Through a draw a parallel to Ob, this will be a tangent to the ellipse. Extend Oa to P, so that Ob is a mean proportional between Oa and OP. Draw a perpendicular bisector of Op intersecting the tangent through a at C. With C as a center draw a circle through P and O, cutting aC in A and B. Draw OA and OB; these will be the semi-axes of the ellipse, for $Aa\times aB=Oa\times aP=Ob^2$. OA and OB are conjugate diameters, and are perpendicular to each other. And these make the required angles with the given diameters. See Salmon's *Conic Sections*, pages 172 and 173.

Also solved by C. N. Schmall.

CALCULUS.

325. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Integrate $\displaystyle\int\frac{4x^6-a^3}{\sqrt{(x^6-a^3)}}dx$.

Solution by MARY E. WILSON, Olivet College, Olivet, Michigan.

Let $x^2=z$. Then $dx=\frac{1}{2}\dfrac{dz}{\sqrt{z}}$. Hence, $\displaystyle\int\frac{4x^6-a^3}{\sqrt{(x^6-a^3)}}=\int\frac{4z^3-a^3}{2\sqrt{(z^4-za^3)}}dz$

$$=\sqrt{(z^4-a^3z)}+C=x\sqrt{(x^6-a^3)}+C.$$

Also solved by G. W. Hartwell, and the Proposer. The Proposer multiplied both numerator and denominator by x, after which the integral is seen directly.

326. Proposed by C. N. SCHMALL, New York City.

Prove $\int_0^\infty \frac{[(\tan^{-1}ax)^2-(\tan^{-1}bx)^2]dx}{x}=\frac{1}{4}\pi^2(\log a-\log b).$

Solution by A. M. HARDING, University of Arkansas, and G. W. HARTWELL, Hamline University, St. Paul, Minnesota.

$$\text{Let } I=\int_0^\infty \frac{(\tan^{-1}ax)^2}{x}dx. \quad \text{Then } \frac{dI}{da}=2\int_0^\infty \tan^{-1}ax.\frac{dx}{\sqrt{(1+a^2x^2)}}$$

$$=\frac{2}{a}\int_0^\infty \tan^{-1}ax.\ \frac{adx}{\sqrt{(1+a^2x^2)}}=\frac{2}{a}.\tfrac{1}{2}(\tan^{-1}ax)^2\Big|_0^\infty=\frac{\pi^2}{4a}.$$

$$\therefore I=\frac{\pi^2}{4}\int_1^a \frac{da}{a}=\tfrac{1}{4}\pi^2\log a.$$

$$\text{Likewise, } J=\int_0^\infty \frac{(\tan^{-1}bx)^2}{x}dx=\frac{\pi^2}{4}\int_1^b \frac{db}{b}=\tfrac{1}{4}\pi^2\log b.$$

$$\therefore \int_0^\infty \frac{[(\tan^{-1}ax)^2-(\tan^{-1}bx)^2]}{x}dx=I-J=\tfrac{1}{4}\pi^2(\log a-\log b).$$

NOTES AND NEWS.

The November, 1912, number of the *Tôhoku Mathematical Journal,* published at Sendai, Japan, contains the following articles: Eine Theorie der linearen Integralgleichungen mit unsymmetrischen Kern, by A. Korn; New proofs of the invariants of the abelian group, by G. A. Miller; On the limits of the roots of an algebraic equation with positive coefficients, by S. Kakeya; A determinant and its application to the theory of homogeneous linear differential equations, by M. Shibayama; On some curves of constant breadth, by M. Fujiwara; Note on Stewart's and Luchterhandt's theorems, by K. Ogura; Un lien géométrique élémentaire. Avec un note du redacteur, by T. Kono; Notes on some points in the differential calculus, by T. Hayashi. This journal is of especial interest because it reflects the rapid mathematical developments in Japan. M.

Professor Slaught, in his "Retrospect and Prospect," has summed up pretty well the past history of the MONTHLY and he also sets forth very clearly what may be expected in the future. We trust that his hopes and aspirations may be fully realized. The personnel of the editorial board, the

energy and enthusiasm of the new Managing Editor, and the influence of the forces that are now giving it financial support, all conspire to presage for the MONTHLY a very useful and successful future. Yet with all the the bright prospects for its future, it cannot render the service designed and must ultimately perish, unless the great army of teachers of mathematics in the high schools and colleges of this country all unite in contributing to its welfare.

It seems to me that it is a sad commentary on the present status of the qualifications of the teachers of mathematics in our high schools, academies, normal schools, and colleges, that so few of them were at any time enrolled on our list of subscribers, and it is no excuse to say that the contributions to the MONTHLY were of too technical a nature to be of interest to these teachers; for, we have it from *School Science and Mathematics* that that journal, at first, fared no better at the hands of the average mathematics teacher. We are glad to know, however, that condititions are changing and that there are progressive forces at work in the world of mathematics teaching as well as in the world of politics and society. On the other hand, in contra distinction to the apathy and lethargy of the teachers of mathematics in high schools, normal schools, and colleges, many of the head professors of mathematics in our leading universities not only secured subscriptions for their university libraries, but were themselves regular subscribers through all the years of its existence. Among many of these, we mention Professor E. H. Moore, of the University of Chicago; Professor W. E. Byerly, of Harvard; Professor Edwin S. Crawley of the University of Pennsylvania; and, while living, Professor H. A. Newton, of Yale, and Irving Stringham, of the University of California.

When this journal was started, nineteen years ago, an urgent appeal was made to the teachers of mathematics in our high schools and colleges to send in contributions and to call for discussions of subjects in which they were specially interested, but, these teachers as a class, failed to respond. We are safe in saying that we have never had, during the nineteen years, a dozen high school teachers of mathematics on our subscription list at any one time. This fact points to what we have many times called attention, viz. the lack of preparation on the part of many of these teachers for the work they are supposed to be doing. Perhaps many of them hide behind the old adage, "Fear the man of few books," and for one, we fear those teachers in any department of knowledge who have few books and who are taking no journal or magazine devoted to their particular line of work. Such we fear and would avoid as we would the bubonic plague, for their infection on the rising generation is no less deadly. A partial remedy for this disease, which, in the end, will work a complete cure, is for college and university professors to refuse to recommend candidates for such positions whose interests in mathematicr do not lie beyond earning a living or social standing. We are teaching because we love it for its own sake. We have spent a great

deal of money for books and magazines. Not every teacher would need to spend as much, but every teacher who has the interest of the growing mind at heart will always keep himself in touch with the very best methods of securing that growth and development; and how can such information be better secured than by taking some live journal devoted to the subject he is teaching?

While we have spoken somewhat deprecatingly of a large number of teachers of mathematics, we have done so not, we hope, in a cynical way. We have simply pointed out a fact which has powerfully impressed itself upon us by personal observation. We believe these conditions will improve in time for already are there signs of better things.

Before bringing these reminiscences to a close, we wish to record our appreciation of a few of the many friends who gave us aid and encouragement in the task of carrying on the publication of the MONTHLY for so many years. It is a fact that the MONTHLY has been continued for nearly twice as long a period as any other mathematical journal published under private control, in this country; and had the reorganization not taken place this year and had Editor Slaught withdrawn, the MONTHLY would have gone on as it has with the hope that some day such an arrangement as the present one would be consummated.

When we had under advisement the publication of the MONTHLY and the object which it was wished to accomplish thereby, our first encouragement came in the way of a cash subscription from that well-known scholarly educator of the Middle West, Superintendent J. M. Greenwood, of Kansas City, Missouri. We are delighted to record the fact that our first subscriber was this eminently successful educator and scholar. We wish here also to speak most appreciatively of one other splendid personage, whom, we fear, will never receive the universal recognition and homage due him in this life, but who (a prophet is not without honor save in his own country) is highly honored and esteemed by the great savants of Europe and the far East. We refer to our good friend, Dr. George Bruce Halsted, who, while professor of mathematics in the University of Texas, shared with us equally for a number of years, the annual deficit, and we are glad, in this public way, to express to him our sincere appreciation for his unselfish devotion and kindly interest in the MONTHLY. Not only did he help the MONTHLY financially, but he also contributed numerous articles which have attracted considerable attention both at home and abroad.

In this connection, we wish to thank our friends, Doctors Dickson, Slaught, and Miller, for the very excellent service they have all rendered to the MONTHLY and through it to the mathematical teaching of this country. As Professor Slaught well says, Dr. Dickson's editorial connection with the MONTHLY was most fortunate. Though only a very young man, his high attainments as a mathematician of the first rank added greatly to the influence of the MONTHLY, and it was with the deepest regret

that we received from him the word that owing to increased duties in the University, he was obliged to give up the extra work which the MONTHLY entailed. His connection with the MONTHLY began with the October number of Volume IX, for a statement of which see that number.

His successor, Professor Slaught, has been no less energetic, and, if anything, even more enthusiastic. As he says, he has worked, and talked, and sought aid for the MONTHLY morning, noon, and night, and by his efforts additional financial aid came to us through the University of Illinois. By this means also, Dr. Miller was added to the editorial staff to the still further great advantage of the MONTHLY.

Also to Mr. J. M. Colaw, who in the early days rendered invaluable assistance in securing subscribers and thus enlarging the MONTHLY'S clientele; and to Dr. Epsteen and Dr. Glenn for their work during our two year's leave of absence from Drury College.

In considering all the agencies which have contributed to the success of the MONTHLY, it would be very unjust not to consider the two printers, who, in these last nineteen years did the printing of the journal. Had it not been for the sacrifices, first, of Mr. Ed. J. Chubbuck, of Kidder, Missouri, and second, of Mr. S. A. Dixon, of Springfield, Missouri, the MONTHLY could never have been founded and continued.

Mr. Chubbuck undertook the publication of the MONTHLY on a basis which brought him for the first volume scarcely $300. He published the first ten numbers of the second volume at about the same rate. When we came to Drury College in 1895, Mr. Dixon joined us in the sacrificial work. We contracted with him for a 32-page issue for $30. But the price of labor and material quickly rose and we were obliged to increase the price of publication, which price always exceeded the income from the gradually increasing subscriptions. We were thus forced to either increase the subscription price or decrease the number of pages per issue. We chose the latter alternative. During all these more than seventeen years, Mr. Dixon never received commercial price for the work he did on the MONTHLY. Being an unusually swift and accurate type-setter himself, he set the type for the MONTHLY with his own hands, and thus was enabled to carry on its publication without serious financial outlay to himself. Could we have paid him commercial rates for his work he would have been able to furnish himself with outside labor and with a larger assortment of type. Under these conditions we do not hesitate to say that he would have turned out his work in the way of typographical neatness and style which could not have been excelled by the most elaborately equipped establishments in the country.

While there has often been exasperating delays in its publication, be it never forgotten that the publication of the MONTHLY has always been a "labor of love" on the part of both printers and editors.

To all these friends, and to Mrs. Finkel, who helped us in the proofreading of nearly every page, as well as to all who have in any way given us

aid and encouragement, we hereby extend our sincerest thanks, and we bespeak for the MONTHLY under the new organization an enlarged and potent field of future usefulness. We invite all our old friends to enlist with us in the new venture.

With kindest greetings of the season, and wishing all a Happy New Year, we remain, Very sincerely,

B. F. FINKEL.

BOOKS.

Elementary Textbook on the Calculus. By Virgil Snyder, Ph. D., and John Irwin Hutchinson, Ph. D., of Cornell University. 8vo. Cloth Sides, Leather Back. 384 pages. Price, $2.00. New York and Chicago: The American Book Co.

In view of the ever-increasing demand to curtail mathematical study as a pure science, for all students taking preprofessional courses, the authors of this text have endeavored to present the calculus in as simple, direct, accurate, and rigorous form as possible. They present the derivative as a limit, an idea easily grasped by the average student, and stimulate the student's interest by applications to maxima and minima, tangents, normals, etc., and they have intentionally avoided making the calculus a treatise on mechanics. The problems and illustrations are suitably chosen and well adapted to the growing mastery of the student. F.

Important Mathematical Discoveries. Part I. Preliminary Demonstrations Leading to Equalizing of Curved Lines to Straight Lines; Part II. Perimeters and Circumferences made Equal to Straight Lines; Part III. The Trisection of an Angle; Part IV. The Quadrature and Duplication of the Cube. By P. D. Woodlock, Columbia, Mo.

The author of this little volume of 39 pages thinks he has added something new to mathematical science, and that he has settled some of those recondite questions that have baffled the mathematical geniuses of the past two thousand years. Being a resident of the home of the Missouri State University, it seems rather strange that the author has not invited a single one of the professors of that institution to confirm the accuracy of his wonderful discoveries. F.

Practical Geometry and Graphics. By David Allan Low (Whitworth Scholar), M. I. Mech. E., Professor of Engineering, East London College (University of London). With over 800 Illustrations and over 700 exercises. 8vo. Cloth, vii+448 pages. Price, $2.50. New York: Longmans, Green & Co.

The field covered by this book is very wide as may be seen from the table of contents. Thus in the Introduction, the author describes some of the instruments used and defines some of the terms. Then follows treatment of the Circle; Conic Sections; Tracing Paper Problems; Approximate Solutions of Some of the Unsolved Problems; Roulettes and Glissettes; Vector Geometry; Graphic Statics; Plane Coördinate Geometry; Periodic Motion; Projection; Projections of Points and Lines; Projections of Simple Solids in Simple Positions; Changing the Plane of Projection; Planes other than the Coördinate Planes; Straight Line and Plane; Sections of Solids; The Sphere, Cylinder, and Cone; Special Projections of Plane Figures and Solids; Horizontal Projections; Pictorial Projections; Perspective Pro-

jections; Curved Surfaces and Tangents; Developments; Helices and Screws; Intersection of Surfaces; Projection of Shadows; Miscellaneous Problems in Solid Geometry; Appendix; Mathematical Tables.

The treatment of all of these subjects is very lucid, and the illustrations are very fine. This is one of the very best books we have yet seen, and will prove an inspiration and joy in the hands of the student under competent instruction. F.

The Calculus. By Ellery Williams Davis, Professor of Mathematics, the University of Nebraska; assisted by William Charles Brenke, Associate Professor of Mathematics, the University of Nebraska; and Edited by Earle Raymond Hedrick, the University of Missouri. 8vo. Cloth, xx+384 pages+63 pages of Tables. Price, $2.00. New York: The Macmillan Co.

This book possesses a number of important features which should commend it to the interested teacher of the calculus. Among these we mention the derivation of Taylor's Series without introducing an extraneous series the reason for which always seems forced and artificial to the student; the omission of certain traditional theorems, and the inclusion of others considered essential both on mathematical and scientific grounds; and the creation of a new word, viz, the word *flexion,* which means the rate of change of the slope with respect to the abscissa. As examples of traditional theorems included in this book are Cavalieri's theorem, the prismoid formula, and the principle of least squares, given under the head of exercises in maxima and minima. The authors have used some new methods for the derivation of sine and logarithm. These methods are certainly interesting, but we have some doubts as to their pedagogical value. In our seventeen years' experience in teaching the calculus, we have always had a number of good students who for some reason had never had a course in physics. To such students, the method of setting up the fundamental principles of the calculus by borrowing ideas from some other science, as, for example, physics, was always unsatisfactory. A teacher using such a book and following the text as a guide would always have to turn aside from the calculus and give a course of two or three lectures on the composition and resolution of velocities—valuable, entertaining, and instructive, to be sure, but wholly impracticable for the teacher whose time at his disposal for the teaching of the calculus, is so very limited. Our own experience has led us to lay great emphasis on the notion of a function and the definition of a derivative, giving many simple illustrations to secure vivid ideas and then derive the derivatives of the various functions in a perfectly straightforward way. Thus

$$\frac{d}{dt}(\sin ax) = \lim_{\triangle t \doteq 0}\left[\frac{\sin a(x+\triangle x)-\sin ax}{\triangle x}\right]\frac{\triangle x}{\triangle t}.$$

The same may be said of the derivative of a logarithm, which method is somewhat out of the ordinary. When the fundamental principles of the calculus are once firmly established, the teacher can wander with his class into any of the closely lying fields of science with pleasure and profit to his students. We recognize, however, that here is good ground for "doctors to disagree." The book is well conceived and full of very interesting material both for students of engineering and of pure mathematics. F.

Theorie der Prym'schen Funktionen Erster Ordung im Anschluss an die Schöpfungen Riemann's. Von Frederich Prym und Georg Rost, mit 25 Figuren im Text. 4to. Three-quarters Leather. Leipzig, Germany: B. G. Teubner.

This volume is divided into two parts. The first part, consisting of xi+250 pages, is devoted to the Foundation of the Theory of Prym's Functions, while the second part, consisting of vi+300 pages, is devoted to the System of Functions.

The first part contains seven chapters and an appendix containing four articles by Prym. The first five chapters are devoted to the integration of the partial differential exuation $\triangle u=0$, under certain limitations, while the sixth deals with the establishment and proof of the fundamental theorems of the theory, and the seventh with the establishment of the fundamental formula. F.